94. Notes on (m, n)-Ideals. I

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Let S be a semigroup. A subsemigroup A of S is called (m, n)-ideal of S, if A satisfies the condition

$$A^m S A^n \subseteq A \tag{1}$$

where m, n are non-negative integers (A^m is suppressed if $m=0$). By a proper (m, n)-ideal we mean an (m, n)-ideal, which is a proper subset of S. The concept of (m, n)-ideal is a generalization of one-sided (left or right) ideals in semigroups and was introduced in [2]. (See also [3], [4], [5], [6] and [1].)

In this note we prove some theorems on (m, n)-ideals.

Theorem 1. *Let S be a semigroup, T be a subsemigroup of S and let A be an (m, n)-ideal of S. Then the intersection $A \cap T$ is an (m, n)-ideal of the semigroup T.*

Proof. The intersection $A \cap T$ evidently is a subsemigroup of S. We show that $A \cap T$ satisfies (1). First, we see that

$$(A \cap T)^m T (A \cap T)^n \subseteq A^m S A^n \subseteq A \tag{2}$$

because of A is an (m, n)-ideal of S. Secondly

$$(A \cap T)^m T (A \cap T)^n \subseteq T^m T T^n \subseteq T \tag{3}$$

therefore (2) and (3) imply

$$(A \cap T)^m T (A \cap T)^n \subseteq A \cap T,$$

that is the intersection $A \cap T$ is an (m, n)-ideal of T.

Theorem 2. *Let S be a semigroup, A be an (m, n)-ideal of S and let B be a subset of S satisfying either $AB \subseteq A$ or $BA \subseteq A$. Then the products AB and BA are (m, n)-ideals of S (m, n are positive integers).*

Proof. Suppose that e.g. the condition $AB \subseteq A$ is fulfilled. Hence

$$(AB)(AB) \subseteq A \cdot AB \subseteq AB,$$

i.e. AB is a subsemigroup of S. On the other hand

$$(AB)^m S (AB)^n \subseteq A^m S A^{n-1} \cdot (AB) \subseteq AB$$

because of A is an (m, n)-ideal of S. Thus AB is an (m, n)-ideal of S.

We prove that BA is also (m, n)-ideal of S. Since

$$(BA)(BA) = B(AB)A \subseteq BA \cdot A \subseteq BA,$$

BA is a subsemigroup of S. From the condition $AB \subseteq A$ it follows, that

$$(BA)^m S (BA)^n \subseteq B \cdot A^m S A^n \subseteq BA,$$

therefore BA is also an (m, n)-ideal of S.

Analogously we can prove our theorem if the condition $BA \subseteq A$

is satisfied.

Corollary. *Let A be an (m, n)-ideal of a semigroup S and let a be an element of A. Then the products aA and Aa are (m, n)-ideals of S (m, n are positive integers).*

This follows at once from Theorem 2.

Theorem 3. *Let S be a semigroup, which satisfies the descending chain condition for its subsemigroups. If S has at least one proper (m, n)-ideal, where $m>1$, $n>1$, then S has either a proper $(1, k)$-ideal or a proper $(k, 1)$-ideal, too.*

Proof. Let m_1 be the smallest positive integer for which there exists proper (m_1, n)-ideal in S, and let n_1 be the smallest positive integer such that there exists proper (m, n_1)-ideal in S. We show that either $m_1 \leqq n$ or $n_1 \leqq m$ holds. If would be $m_1 > n$ and $n_1 > m$, then $m_1 \leqq m$ implies $n < n_1$, which is impossible.

Suppose that $1 < m_1 \leqq n$ and A is a proper (m_1, n)-ideal of S. We define the following sequence of subsemigroups of S:

(4) $$B_1 = A^{m_1} S A^n; \quad B_{i+1} = B_i^{m_1} S B_i^n, \quad (i=1,2,\cdots).$$

It is easy to see, that

(5) $$B_i^{m_1} S B_i^n \subseteq A \quad (i=1,2,\cdots)$$

holds. From the descending chain condition for subsemigroups of S it follows, that there exists a positive integer j such that

$$B_j = B_{j+1},$$

that is

$$B_j = B_j^{m_1} S B_j^n.$$

We shall write B instead of B_j. Therefore

(6) $$B = B^{m_1} S B^n.$$

This implies

(7) $$B^{m_1} S B^{n-m_1} B^{m_1} S B^n = B S B^n,$$

and

(8) $$B^{m_1} S B^{n-m_1+1} = B S B^n.$$

From (6) and (8) we conclude that

$$B^{m_1} S B^{n-m_1+1} \cdot B^{m_1-1} = B S B^n \cdot B^{m_1-1},$$

that is

$$B S B^{n+m_1-1} = B.$$

Thus the subsemigroup B is an $(1, n+m_1-1)$-ideal of S.

Analogously we can prove the existence of proper $(k, 1)$-ideal of S in case of $n_1 \leqq m$.

References

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