146. Boolean Elements in Lukasiewicz Algebras. I

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0. *INTRODUCTION.* In the theory of the (three-valued) Lukasiewicz algebras founded by Gr. Moisil, the possibility operator plays an important role. Moisil denotes the operator by M and we shall denote by ∇ it defined on a distributive lattice A and it is uniquely determined by the set K of all elements $k\in A$ such that $\nabla k=k$.

The purpose of this note is to establish characteristic properties of the family K. In § 1 we summarize some theorems on closure operators defined on lattices. In § 2, we study these operators in the case of Kleene algebras, and in § 3 we apply these results to the problem suggested by A. Monteiro.*)

1. *CLOSURE LATTICES.* Let $(L, 0, 1, \wedge, \vee)$ be a lattice with first and last elements. If a unary operator ∇ is defined on L such that:

$C\,1)$ $\nabla 0=0$, $C\,2)$ $x\leq \nabla x$,

$C\,3)$ $\nabla(x\vee y)=\nabla x\vee \nabla y$, $C\,4)$ $\nabla\nabla x=\nabla x$,

we shall say that the system $(L, 0, 1, \wedge, \vee, \nabla)$ is a *closure lattice*, and the operator ∇ is a *closure operator*. This notion is a generalization of closure operators on topological spaces and was studied by N. Nakamura [17] (see also [16] and [18]).

It is easy to prove that:

$C\,5)$ If $x\leq y$, then $\nabla x\leq \nabla y$, or equivalently,

$C\,6)$ $\nabla(x\wedge y)\leq \nabla x\wedge \nabla y$.

In [18] it was proved that

1.1. *The family K of all invariant elements of a closure operator has the following properties*:

$K\,1)$ *K is a sub-lattice of L containing* 0 *and* 1.

$K\,2)$ *K is lower relatively complete: that is, for all $x\in L$, the set $\{k\in K : x\leq k\}$ has an infimum belonging to K.*

Moreover we have

$$\nabla x=\wedge\{k\in K : x\leq k\}. \qquad (1)$$

Conversely, if K is a subset of L with the properties $K\,1)$ *and* $K\,2)$, (1) *defines a closure operator ∇ on L, and K is the set of all invariant elements by ∇.*

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We shall say that a unary operator Δ defined on L satisfying

I 1) $\Delta 1=1$, I 2) $\Delta x\leq x$,

I 3) $\Delta(x\wedge y)=\Delta x\wedge\Delta y$, I 4) $\Delta\Delta x=\Delta x$

is an *interior operator*.

In [18] the dual form of 1.1. was also proved:

1.2. *The family H of all invariant elements of an interior operator Δ has the following properties*:

H 1) *H is a sub-lattice of L containing* 0 *and* 1.

H 2) *H is upper relatively complete: that is, for all $x\in L$ the set $\{h\in H: h\leq x\}$ has a supremum belonging to H.*

Moreover we have

(2) $$\Delta x=\bigvee\{h\in H: h\leq x\}.$$

Conversely, If H is a subset of L with the properties H 1) and H 2), (2) defines an interior operator Δ on L, and H is the set of all invariant elements by Δ.

2. *KLEENE ALGEBRAS.* Let $(A,\wedge,\vee)$ be a distributive lattice. If a unary operation $\sim$ is defined on A such that:

M 1) $\sim\sim x=x$, M 2) $\sim(x\vee y)=\sim x\wedge\sim y$,

we shall say that the system $(A,\wedge,\vee,\sim)$ is a *de Morgan lattice*. This notion has been introduced by Gr. Moisil ([11], p. 91) and studied by J. Kalman [7] under the name of *distributive i-lattice*. It is easy to prove that $\sim$ is an *involution* ([4], p. 4), that is, it satisfies M1) and

M 3) $x\leq y$ if and only if $\sim y\leq\sim x$.

As $\sim$ is an involution, we have that if $\{x_i\}_{i\in I}$ is a family of elements of A such that $\bigvee_{i\in I} x_i$ exists, then $\bigwedge_{i\in I}\sim x_i$ also exists and we have

M 4) $$\sim\bigvee_{i\in I} x_i=\bigwedge_{i\in I}\sim x_i.$$

Analogously, if $\bigwedge_{i\in I} x_i$ exists, then $\bigvee_{i\in I}\sim x_i$ also exists and

M 5) $$\sim\bigwedge_{i\in I} x_i=\bigvee_{i\in I}\sim x_i.$$

If A has the last element 1, we shall say that A is a *de Morgan algebra*. This notion has been studied by A. Bialynicki-Birula and H. Rasiowa ([3], [2]) under the name of *quasi-Boolean algebras*. In this case, $0=\sim 1$ is the first element of A.

If the operation $\sim$ also verifies the condition

K) $$x\wedge\sim x\leq y\vee\sim y,$$

we shall say that A is a *Kleene lattice* (*algebra*). A three-element algebra of this kind was used by S. C. Kleene as a characteristic matrix of a propositional calculus ([8], [9], p. 334). These lattices were studied by J. Kalman [7] with the name of *normal distributive i-lattices*. An important example of Kleene algebras are the N-lattices of H. Rasiowa [19]. We have used the terminology introduced in [15] and [5].

Let A be a de Morgan algebra. We shall say that a *sub-algebra B of A is lower (upper) relatively complete*, if B has property $K\,2)$ of 1.1 (property $H\,2)$ of 1.2.).

From $M\,3)$, $M\,4)$, and $M\,5)$ we can easily prove the following

2.1. *LEMMA. A sub-algebra B of a de Morgan algebra A is lower relatively complete if and only if it is upper relatively complete. In this case the operators ∇ and Δ respectively defined by* (1) *of* 1.1. *and* (2) *of* 1.2. *are related by the following formulae*:

(1) $\Delta x = \sim\nabla\sim x$, (2) $\nabla x = \sim\Delta\sim x$.

We shall say that $x \in A$ is a *Boolean element* if there exists an element $-x \in A$ such that $x \wedge -x = 0$ and $x \vee -x = 1$. We know that if it exists, $-x$ is unique, and will be called the *Boolean complement* of x. Let B be the set of all Boolean elements of A. Clearly B is a Boolean algebra.

We shall use the following result*) by A. Monteiro. For completeness, we give the proof.

2.2. *LEMMA. Let A be a Kleene algebra. If $z \in A$ has a Boolean complement $-z$, then $-z = \sim z$.*

PROOF: By hypothesis we have

(1) $z \vee -z = 1$, (2) $z \wedge -z = 0$,

therefore by $M\,2)$

(3) $\sim z \wedge \sim -z = 0$, (4) $\sim z \vee \sim -z = 1$,

this means

(5) $-\sim z = \sim -z$, (6) $-\sim -z = \sim z$,

and so $\sim z$ and $\sim -z$ are also Boolean elements. By $K)$ we can write

(7) $$z \wedge \sim z \leq -z \vee \sim -z.$$

As z, $-z$, $\sim z$, $\sim -z$ are Boolean elements, so are $(z \wedge \sim z)$ and $(-z \vee \sim -z)$. Then by (7) we have $-(-z \vee \sim -z) \leq -(z \wedge \sim z)$, that is $z \wedge -\sim -z \leq -z \vee -\sim z$, hence, by (5) $z \wedge \sim z \leq -z \vee -\sim z$.

From this relation we deduce

$$z \wedge \sim z \leq (-z \vee -\sim z) \wedge \sim z = (-z \wedge \sim z) \vee (\sim z \wedge -\sim z),$$

hence, by (3) and (5), $z \wedge \sim z \leq -z \wedge \sim z$ and then

$$z \wedge \sim z \leq z \wedge -z \wedge \sim z = 0.$$

So, $z \wedge \sim z = 0$ and by $M2)$, $z \vee \sim z = 1$, which proves $\sim z = -z$.

Q.E.D.

2.3. *COROLLARY. The set B of all Boolean elements of a Kleene algebra A is a subalgebra of A.*

3. (*THREE-VALUED*) *LUKASIEWICZ ALGEBRAS*. The notion of (three-valued) Lukasiewicz algebra was introduced and developed by Gr. Moisil ([12], [13], [14]) to study the three-valued logic of J. Lukasiewicz [10]. Its role is similar to Boolean algebras in

*) Unpublished.

classical logic. We shall use the following A. Monteiro's definition [6] that is equivalent to Gr. Moisil's:

3.1. *DEFINITION. A (three-valued) Lukasiewicz algebra is a system* $(A, 1, \wedge, \vee, \sim, \nabla)$ *such that* $(A, 1, \wedge, \vee, \sim)$ *is a Kleene algebra and* ∇ *is a unary operator defined on* A *that satisfies the following axioms*:*)

$L\,1)$ $\nabla(x \wedge y) \leq \nabla x \wedge \nabla y$, $L\,2)$ $\sim x \vee \nabla x = 1$,

$L\,3)$ $x \wedge \sim x = \sim x \wedge \nabla x$.

Let us recall some properties ([12]-[14]):

$L\,4)$ $\nabla(x \wedge y) = \nabla x \wedge \nabla y$, $L\,5)$ $\nabla(x \vee y) = \nabla x \vee \nabla y$,

$L\,6)$ $x \leq \nabla x$, $L\,7)$ $\nabla\nabla x = \nabla x$,

$L\,8)$ $\nabla x = x$ if and only if x is a Boolean element of A,

$L\,9)$ $\nabla 0 = 0$.

We must notice that the (three-valued) Lukasiewicz algebras are examples of Kleene algebras where a non-trivial operator satisfying $L\,4)$, $L\,6)$, $L\,7)$, and $L\,9)$ is defined, unlike Boolean algebras, where G. Bergman [1] proved that the identity operator is the only one that satisfies such conditions.

Properties $L\,5)$, $L\,6)$, $L\,7)$, and $L9)$ show that ∇ is a closure operator on A, hence according to 1.1 and $L8)$, it follows that the subalgebra B of all Boolean elements of A is lower relatively complete, and for all $x \in A$ we have

$L\,10)$ $$\nabla x = \wedge\{b \in B : b \leq x\}.$$

According to 2.1 and 2.3 we can define the operator Δ, (that is interpreted as the *necessity operator* and noted as ν by Moisil) dual of ∇, by the formula:

$L\,11)$ $$\Delta x = \vee\{b \in B : x \leq b\}$$

and we have the relations (1) and (2) of 2.1.

Moisil proved the following *determination principle* [12]:

$L\,12)$ $x \leq y$ if and only if $\Delta x \leq \Delta y$ and $\nabla x \leq \nabla y$.

From $L10)$, $L11)$, and $L12)$ we easily see that the subalgebra B is *separating*, that is, if $y \not\leq x$ for $x, y \in A$ then, there exists $b \in B$ such that $x \leq b$ and $y \not\leq b$ or there exists $b' \in B$ such that $b' \leq y$ and $b' \not\leq x$.

In short, we can assert that *the family of invariant elements of the operator* ∇ *coincides with the subalgebra of all Boolean elements of* A, *that is lower relatively complete and separating.* The next theorem shows that these properties characterize the set of invariant elements of ∇.

3.2. *THEOREM. Let* A *be a Kleene algebra such that the family* B *of its Boolean elements is lower relatively complete and separating. Then one and only one (three-valued) Lukasiewicz*

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*) The operation $\sim$ was noted as N by Moisil.

algebra structure can be defined on A.

PROOF: As B is lower relatively complete, by the formula L 10), we can define the operator ∇ and according to 1.1, ∇ will have the properties C 1)–C 6) and B will be the family of all invariant elements of ∇. To prove the theorem it is sufficient to show that ∇ also satisfies axioms L 2) and L 3).

Let us prove L 2). By C 2), $x \leq \nabla x$, by M 3) it follows that $\sim \nabla x \leq \sim x$. As ∇x is a Boolean element, from 2.2 it follows that $\sim \nabla x$ is the Boolean complement of ∇x. Hence we have

$$1 = \sim \nabla x \vee \sim x \leq \sim x \vee \nabla x \leq 1.$$

Let us prove L 3). First of all by 1.2, 2.1, and 2.3 the operator Δ can be defined, and will have properties I 1)–I 4), moreover it satisfies

(1) $$\Delta x = \sim \nabla \sim x.$$

As $x \leq \nabla x$, it is clear that $\sim x \wedge x \leq \sim x \wedge \nabla x$, then, to prove $L3$) we need to show

(2) $$\nabla x \wedge \sim x \leq \sim x \wedge x.$$

This last proof will be done in two steps:

I. Let us prove the following property:

(P) If x, y of A satisfy

(P 1) $y \leq \sim x$,

(P 2) for all $b \in B$ such that $x \leq b$ we have $y \leq b$,

then $y \leq x$.

For, let us suppose that $x, y \in A$, x, y satisfy P 1) and P 2) and $y \not\leq x$. As there cannot exist $b \in B$ such that $x \leq b$ and $y \not\leq b$ by P 2), from the separation property of B it follows that there exists $b' \in B$ such that $b' \leq y$ and $b' \not\leq x$. By $b' \not\leq x$, we have in particular

(3) $$b' \neq 0.$$

Moreover

(4) $$b' \leq y \leq \sim x.$$

$\nabla x \in B$ and C 2) imply $x \leq \nabla x$. By P 2), we have $y \leq \nabla x$ and $b' \leq y$, hence

(5) $$b' \leq \nabla x.$$

From (4) and (5)

(6) $$b' \leq \sim x \wedge \nabla x.$$

Applying Δ to both sides of (6) and recalling 1.2 and the formula (1), we have by 2.2

$$\Delta b' = b' \leq \Delta(\sim x \wedge \nabla x) = \Delta \sim x \wedge \Delta \nabla x = \Delta \sim x \wedge \nabla x$$
$$= \sim \nabla x \wedge \nabla x = 0.$$

Then $b' = 0$, which contradicts (3). Therefore we have $y \leq x$, and (P) is proved.

II. (P) and the lower relatively completness of B imply (2). For, making $y=\sim x\wedge \nabla x$ we have

(7) $$y\leq \sim x.$$

Therefore x, y satisfy P 1). They also satisfy P 2). For, if $b\in B$ and $x\leq b$, then $\nabla x\leq \nabla b=b$ so $y\leq \nabla x\leq b$. Then by (P), we have

(8) $$y=\sim x\wedge \nabla x\leq x.$$

From (7) and (8) we have (2). Q.E.D.

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