9. On Mappings in Uniform Spaces over a Topological Semifield

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1. Introduction. In their articles [1] and [2], M. Ya. Antonovskii, V. G. Boltyanskii, and T. A. Sarymsakov have developed to the theory of the topological semifield.

In this paper which is based on theories given at the [1] and [2] we shall prove a theorem similar to that of T. A. Brown and W. W. Comfort (see [3]) in uniform spaces over a topological semifield.

The notation used here very closely approximates that of [1], [2], and [3].

2.1. Proposition. *Let (X, ρ, E) be a metric space over a topological semifield E. For each neighborhood of zero U in E, let $U^*=\{(x, y) \mid \rho(x, y) \in U\}$. Then the family $\mathfrak{U}^*$ of all sets of the form U^* determines a basis for the uniform space over a topological semifield E.*

2.2. Definition. Let $\mathfrak{U}$ be a basis of neighborhoods of zero for a topological semifield E, then $\mathfrak{U}$ is said to be ample if, whenever $x \in U \in \mathfrak{U}$, there is a $W \in \mathfrak{U}$ for which $x \in W \subset \overline{W} \subset U$.

2.3. Definition. Let $\mathfrak{U}^*$ be a basis for the uniform space over the topological semifield E and let f be a function on X into X. Then

(a) f is said to be a contraction with respect to $\mathfrak{U}^*$ if $(fx, fy) \in U^*$ whenever $(x, y) \in U^* \in \mathfrak{U}^*$;

(b) f is said to be an expansion with respect to $\mathfrak{U}^*$ if $(x, y) \in U^*$ whenever $(fx, fy) \in U^* \in \mathfrak{U}^*$;

(c) f is said to be isobasic with respect to $\mathfrak{U}^*$ if f is both a contraction with respect to $\mathfrak{U}^*$ and an expansion with respect to $\mathfrak{U}^*$.

3.1. Theorem. *Let $\mathfrak{U}$ be an open basis of neighborhoods of zero for the totally bounded Hausdorff metric space (X, ρ, E) and $\mathfrak{U}^*$ is a basis for the uniform space over the topological semifield E. If a function f, mapping X onto X, is a contraction with respect to $\mathfrak{U}^*$ and $\mathfrak{U}$ is ample then f is isobasic with respect to $\mathfrak{U}^*$.*

Proof. Let $(fx, fy) \in U^*$, since $\mathfrak{U}$ is ample, we find $W \in \mathfrak{U}$ such that $\rho(fx, fy) \in W \subset \overline{W} \subset U$. Suppose now that $(x, y) \notin U^*$. Then there is a symmetric $V_1^* \in \mathfrak{U}^*$ for which $(x, y) \notin V_1^* \circ W^* \circ V_1^*$. In fact, $\rho(x, y) \notin U$ and $\rho(x, y) \in E \backslash U \subset E \backslash \overline{W}$, the set $E \backslash \overline{W}$ is open and there is a symmetric neighborhood of zero V_1 such that $\rho(x, y)+V_1+V_1 \subset E \backslash \overline{W}$. Since $E \backslash \overline{W} \subset E \backslash W$, we have $\rho(x, y)+V_1+V_1 \subset E \backslash W$, i.e,

$$\rho(x, y)+V_1+V_1 \notin W, \rho(x, y) \notin W-V_1-V_1=V_1+W+V_1.$$

Thus $(x, y) \notin V_1^* \circ W^* \circ V_1^*$.

We now denote by $\Omega(x, U)$ the set of all elements $y \in X$ which satisfy the condition $\rho(x, y) \in U$. Next we choose $V_2 \in \mathfrak{U}$ so that $\Omega(fx, V_2) \times \Omega(fy, V_2) \subset W^*$ and select $V \in \mathfrak{U}$ so that $V \cup V^{-1} \subset V_1 \cap V_2$. If $\rho(x, x') \in V$ and $\rho(y, y') \in V$ than $\rho(x', y') \notin W$ and $\rho(fx', fy') \in W$.

Now, let $A=\{a_1, a_2, \cdots, a_n\}$ be a minimal V-net for which the number of the set (a_i, a_j) such that $\rho(a_i, a_j) \in W$ for $i \neq j$ is maximal. Since there is no index i for which $\rho(x, a_i) \in V$ and $\rho(y, a_i) \in V$, we may suppose that $\rho(x, a_1) \in V$ and $\rho(y, a_2) \in V$. Then $\rho(a_1, a_2) \notin W$. Next, let $fA=\{fa_1, fa_2, \cdots, fa_n\}$. Then fA is a minimal V-net and $\rho(fa_1, fa_2) \in W$. Moreover, $\rho(fa_i, fa_j) \in W$ whenever $\rho(a_i, a_j) \in W$. This contraction completes the proof.

References

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