28. *Minimal or Smallest Relation of Given Type*

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1. This note is to announce the improvement and development of the results in [4] and also to report a brief note of a part of [5]. The earlier paper [4] discussed the smallest of the relations of a given type which contain a given relation, and as its application, the smallest congruence of certain type. In this note we shall treat minimal or smallest relation of given types in the most general cases.

Following Birkhoff [1], we define minimal (smallest) element and maximal (greatest) element in a partially ordered set E with an ordering $\leqq$.

An element a of E is called a minimal (maximal) element of E if

$$x \in E,\ x \leqq a\ (x \geqq a) \text{ imply } x = a.$$

An element a of E is called a smallest (greatest) element of E if

$$x \geqq a\ (x \leqq a) \text{ for all } x \in E.$$

Let S be a set and let $\rho, \sigma \cdots$ denote binary relations on S i.e. subsets of $S \times S$. Let $\mathscr{B}_0$ be the set of all binary relations (which we shall call "relations") on S. $\mathscr{B}_0$ is a complete lattice with respect to inclusion where the empty relation $\square$ is smallest and the universal relation $\omega = S \times S$ is greatest.

2. A subset $\mathscr{T}$ of $\mathscr{B}_0$ is called a "pretype" of relations on S or briefly a pretype on S if $\mathscr{T}$ shall contain $\square$ and if $\mathscr{T}$ is a non-empty subset of $\mathscr{B}_0$; a pretype $\mathscr{T}$ is called a "type" of relations if a pretype $\mathscr{T}$ contains ω. A type $\mathscr{T}$ is called a "basic type" if a type $\mathscr{T}$ satisfies the following condition: for any subset $\{\rho_\xi\}$ of $\mathscr{T}$, the intersection $\bigcap_\xi \rho_\xi \in \mathscr{T}$. A basic type is a complete lattice contained in $\mathscr{T}$ but not necessarily sublattice of $\mathscr{B}_0$. Each relation ρ belonging to a pretype $\mathscr{T}$ is called a $\mathscr{T}$-relation on S.

Let $\mathscr{T}$ be a pretype on S and let σ be a non-empty element of $\mathscr{T}$. An element ρ_σ of $\mathscr{T}$ is called a minimal $\mathscr{T}$-relation containing σ if ρ_σ is a minimal element of the set of all relations $\rho(\in \mathscr{T})$ which contain σ; ρ_σ is called the $\mathscr{T}$-relation generated by σ if ρ_σ is a smallest element of the set of all relations $\rho(\in \mathscr{T})$ which contain σ. An element ρ_0 of $\mathscr{T}$ is called a minimal $\mathscr{T}$-relation on S or we say S has a minimal $\mathscr{T}$-relation if ρ_0 is a minimal element of $\mathscr{T} \backslash \square$ which denotes the set of all non-empty elements of $\mathscr{T}$. An element ρ_0 is called a smallest $\mathscr{T}$-relation on S or we say S has a smallest $\mathscr{T}$-relation if ρ_0 is a smallest element of $\mathscr{T} \backslash \square$.

Theorem 1. *The intersection of arbitrary number of basic types is also a basic type. Any type is a set union of basic types.*

Let $\mathcal{T}$ be a pretype of relations on S. A type $\overline{\mathcal{T}}$ is defined as follows:

$$\overline{\mathcal{T}}=\begin{cases}\mathcal{T}\cup\{\omega\} & \text{if } \omega\notin\mathcal{T}\\ \mathcal{T} & \text{if } \omega\in\mathcal{T}\end{cases}$$

S has a smallest (minimal) $\mathcal{T}$-relation if and only if S has a smallest (minimal) $\overline{\mathcal{T}}$-relation. Therefore we can restrict $\mathcal{T}$ to types throughout our arguments.

3. Consider a unary operation P of $\mathcal{B}_0$ into itself: $\rho\in\mathcal{B}_0$, $\rho\to\rho P$. P is called a semi-closure operation on $\mathcal{B}_0$ if P is isotonic and extensive [4], and an idempotent semi-closure operation is called a closure operation on $\mathcal{B}_0$ [4]. Let $\mathfrak{P}$ be the set of all semi-closure operations on $\mathcal{B}_0$. $\mathfrak{P}$ is a complete lattice with respect to an ordering $P\leqq Q$ namely $\rho P\subseteq\rho Q$ for all $\rho\in\mathcal{B}_0$ if we admit an empty operation (see [4]). For any semi-closure operation P on $\mathcal{B}_0$, the set of all closure operations which contain P has a smallest element $\bar{P}$. $\bar{P}$ is called the closure operation generated by P. Basic types are closely related to semi-closure or closure operations on $\mathcal{B}_0$ as follows:

Theorem 2. *A type $\mathcal{T}$ on S is a basic type if and only if there is a semi-closure operation P on $\mathcal{B}_0$ such that $\mathcal{T}=\mathcal{B}_0\bar{P}$. There is a one-to-one correspondence between all the closure operations P on $\mathcal{B}_0$ and all the basic types $\mathcal{T}_P$ on S under a mapping $P\to\mathcal{T}_P$, $\mathcal{T}_P=\mathcal{B}_0P$, such that for arbitrary number of closure operation P_ξ on $\mathcal{B}_0$,*

$$\mathcal{B}_0\Big(\bigcap_{\xi} P_\xi\Big)=\bigcap_{\xi}\mathcal{T}_{P_\xi}.$$

Thus a basic type $\mathcal{T}$ is characterized by a semi-closure operation P or a closure operation $\bar{P}$. $\mathcal{T}$ is denoted by $\mathcal{T}\sim P$ or $\mathcal{T}\sim\bar{P}$. Let $\{P_\xi;\ \xi\in\Xi\}$ be a family of semi-closure operations on $\mathcal{B}_0$. A relation ρ will be said to be of type $\bigwedge_{\xi} P_\xi$ if $\rho P_\xi=\rho$ for all $\xi\in\Xi$, and of type $\bigvee_{\xi} P_\xi$ if $\rho P_\xi=\rho$ for some $\xi\in\Xi$. The set of all relations ρ of type $\bigwedge_{\xi} P_\xi$ is a basic type. However the set of all relations of type $\bigvee_{\xi} P_\xi$ is not necessarily a basic type; we call it a "join type" if $|\Xi|>1$. Any type $\mathcal{T}$ is expressed by $\mathcal{T}\sim\bigvee_{\xi\in\Xi} P_\xi$, $|\Xi|\geqq 1$; it is a basic type if $|\Xi|=1$. It is preferable that the terminology "meet type" in [4] is replaced by "basic type".

4. According to [4], for any basic type $\mathcal{T}$ and for any element $\rho\in\mathcal{B}_0$, there is the $\mathcal{T}$-relation generated by ρ; for a type $\mathcal{T}\sim\bigvee_{\xi\in\Xi} P_\xi$, $|\Xi|>1$, Theorem 4.4 in [4] gives a necessary and sufficient condition for the $\mathcal{T}$-relation generated by ρ to exist. Here we give a few theorems related to a minimal $\mathcal{T}$-relation containing ρ and a minimal

or smallest $\mathcal{T}$-relation.

Theorem 3. *Let $\mathcal{T}\sim\bigvee_{\xi\in\Xi} P_\xi$, $|\Xi|>1$. There exists a minimal $\mathcal{T}$-relation containing a $\rho\in\mathcal{B}_0$ if and only if the set $\{\rho P_\xi;\xi\in\Xi\}$ contains at least one minimal element, say ρP_{ξ_0}. Then ρP_{ξ_0} is a minimal $\mathcal{T}$-relation containing ρ.*

Let $\mathcal{T}\sim\bigvee_{\xi\in\Xi} P_\xi$, $|\Xi|\geqq 1$, in Theorems 4, 5, 6 below where if $|\Xi|=1$, $\mathcal{T}$ is a basic type. Let $\mathcal{A}$ be the set of all minimal elements of $\mathcal{B}_0\backslash\square$.

Theorem 4. *The following statements are equivalent.*

(4.1) *S has a minimal $\mathcal{T}$-relation.*

(4.2) *The set $\{\rho\bar{P}_\xi;\rho\in\mathcal{A}\cup\{\square\},\xi\in\Xi\}$ contains its minimal element.*

(4.3) *The set $\{\square\bar{P}_\xi;\xi\in\Xi\}$ contains its minimal element.*

Theorem 5. *The following statements are equivalent.*

(5.1) *S has the smallest $\mathcal{T}$-relation.*

(5.2) *The set $\{\rho\bar{P}_\xi;\rho\in\mathcal{A},\xi\in\Xi\}$ contains its smallest element.*

(5.3) $\bigcap\{\rho\bar{P}_\xi;\rho\in\mathcal{A},\xi\in\Xi\}\neq\square$.

(5.4) $\bigcap_\xi\square\bar{P}_\xi\neq\square$.

5. We can define pretypes, types, and basic types of congruence relations. These are regarded as subsets of the set C of all congruence relations on S. One proposes a question how the semi-closure operations on C are related to the semi-closure operations on $\mathcal{B}_0$. This question is solved in the following way from the more general point of view:

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two complete lattice of relations in which the join, meet, ordering in $\mathcal{B}_i$ are respectively denoted by $\cup_i$, $\cap_i$, $\underset{i}{\subseteq}$ $(i=1,2)$. $\mathcal{B}_1$ is called order-invariant in $\mathcal{B}_2$ if the following two conditions are satisfied

(6.1) $\mathcal{B}_1$ is a subset of $\mathcal{B}_2$.

(6.2) If $\rho,\sigma\in\mathcal{B}_1$ then $\rho\underset{1}{\subseteq}\sigma$ if and only if $\rho\underset{2}{\subseteq}\sigma$.

$\mathcal{B}_1$ is called meet-invariant in $\mathcal{B}_2$ if (6.2) is replaced by (6.3) below:

(6.3) If $\rho_\xi\in\mathcal{B}_1$, $\xi\in\Xi$, then $\bigcap_{1\,\xi}\rho_\xi=\bigcap_{2\,\xi}\rho_\xi$.

Let $\mathfrak{P}_i$ be the complete lattice of all semi-closure operations on $\mathcal{B}_i$ $(i=1,2)$ and $\mathfrak{P}_{\mathcal{B}_1}$ the set of all semi-closure operations Q on $\mathcal{B}_2$ such that $\mathcal{B}_1Q\underset{1}{\subseteq}\mathcal{B}_1$.

Theorem 6. *Suppose that $\mathcal{B}_1$ is order-invariant in $\mathcal{B}_2$. Then the restriction of any Q in $\mathfrak{P}_{\mathcal{B}_1}$ to $\mathcal{B}_1$ is a semi-closure operation on $\mathcal{B}_1$, and any semi-closure operation on $\mathcal{B}_1$ is obtained in this manner. $\mathfrak{P}_{\mathcal{B}_1}$ contains a lattice $\mathfrak{P}_1'$ (not necessarily a sublattice of $\mathfrak{P}_2$) which is the homomorphic image of $\mathfrak{P}_{\mathcal{B}_1}$ as partially ordered semigroups and which is isomorphic onto $\mathfrak{P}_1$ as lattices and as partially ordered semigroups.*

Let C be the set of all quasi-orderings or of all equivalences on

S or of all congruences on S if S is an algebraic system. We have a corollary to Theorems 4, 5:

Corollary. *Let* $\mathscr{T}\sim\bigvee_{\xi\in\Xi}P_\xi$, $|\Xi|\geqq 1$, *be a type as a subset of* C *on* S. *Then* S *has a minimal (smallest)* $\mathscr{T}$*-quasi-ordering or* $\mathscr{T}$*-equivalence or* $\mathscr{T}$*-congruence if and only if the set* $\{\iota\bar{P}_\xi;\ \xi\in\Xi\}$ *has a minimal (smallest) element where* ι *is the equality relation.*

6. We give a few examples to have this note understood.

Example 1. Let S be any set, $\mathscr{T}$ be a basic type given by the property "symmetry and transitivity". S has no smallest $\mathscr{T}$-relation but S has minimal $\mathscr{T}$-relations, $\{(x, x)\}$ consisting of a single element (x, x), $x\in S$.

Example 2. Let S be the set of all positive real numbers with a usual ordering $\leqq$. For each $\rho\in\mathscr{B}_0$, ρP is defined as follows:

$$\rho P=\{(x, y);\ x\leqq a,\ y\leqq b,\ \text{for some } (a, b)\in\rho\}.$$

P is a closure operation on $\mathscr{B}_0$. Let $\mathscr{T}\sim P$. S has no minimal $\mathscr{T}$-relation.

Example 3. Let S be the direct product of an abelian group and a right zero semigroup (cf. [2]). Let $\mathscr{T}$ be the set of all congruence ρ on S such that S/ρ is either commutative or idempotent. S has no smallest $\mathscr{T}$-congruence but has minimal $\mathscr{T}$-congruence.

Example 4. Let S be the semigroup of all positive integers with usual addition. Let $\mathscr{T}$ be given by the property that S/ρ has a zero. S has no minimal $\mathscr{T}$-congruence.

In the sections 1 through 5 we have treated the types of relations most abstractly and most generally. The concept of types depends on an individual set or algebraic system S. However the types of congruences ρ on S are frequently given in terms of the properties of S/ρ. N. Kimura [3] contributed much to this problems. In such a case as identities or implications, we can introduce the types independently of individual S, but dependently on a class of S.

References

[1] G. Birkhoff: Lattice Theory, rev. ed., Amer. Math. Soc. Colloq. Publi., Vol. 25. Amer. Math. Soc., Providence R. I. (1948).

[2] A. H. Clifford and G. B. Preston: Algebraic Theory of Semigroups Vol. 1, Math. Surveys No. 7. Amer. Math. Soc., Providence R. I. (1961).

[3] N. Kimura: On some existence theorems on multiplicative system I. Greatest quotient. Proc. Japan Acad., **34**, 305-309 (1958).

[4] T. Tamura: Operation on binary relations and their application. Bull. Amer. Math. Soc., **70**, 113-120 (1964).

[5] ——: Theory of operations on binary relations (to appear in Trans. Amer. Math. Soc.).