46. *On the Completeness of the Leibnizian Modal System with a Restriction*

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§ 1. **Introduction.** The purpose of this paper is to show the completeness of a modal system which will be called L_0 in the following.

In my previous paper [1], in order to show an example of defence of circular definition, the following definition was given:

A statement is *analytic* if and only if it is *consistent* with every statement that expresses what is *possible*.

This definition, roughly speaking, is materially equivalent to Carnap's definition of L-truth which is suggested by Leibniz' conception that a necessary truth must hold in all possible worlds (cf. Carnap [2], p. 10).

If "analytic" is replaced by "necessary" in the above definition, this definition becomes as follows:

A statement is *necessary* if and only if it is *consistent* with every statement that expresses what is *possible*.

This reformed definition is symbolized by modal signs as follows:

$$\Box p \equiv (q)[\Diamond q \supset \Diamond(p \cdot q)],$$

where p and q are propositional variables.

Let us replace it by the following axiom-schema and rule:

Axiom-schema. $\Box\alpha \supset [\Diamond\beta \supset \Diamond(\alpha\cdot\beta)]$, where α, β are arbitrary formulas.

Rule of inference. If $\vdash \Diamond p \supset \Diamond(\alpha\cdot p)$, then $\vdash \Box\alpha$, where α is an arbitrary formula and p is a propositional variable not contained in α.

We call L (the Leibnizian modal system) the system obtained from the usual propositional calculus by adding the above axiom schema and rule and the rule of replacement of material equivalents. ($\Diamond\alpha$ is regarded as the abbreviation of $\sim\Box\sim\alpha$).

This system is easily proved to be equipollent to the system obtained from the usual propositional calculus by adding the following axiom-schema and rule and the rule of replacement:

Axiom-schema. $\Box(\alpha\supset\beta)\supset(\Box\alpha\supset\Box\beta)$.

Rule. If α is a tautology, then $\vdash\Box\alpha$.

We call L_0 the latter system with the restriction that if $\Box\alpha$ is a formula of L_0 then α does not contain $\Box$. We shall discuss the completeness of L_0 in the following sections.

§ 2. **Main results.** We write $\alpha, \beta, \gamma, \cdots$ for the formulas of L_0

which do not contain $\Box$. We write $\alpha', \beta', \gamma', \cdots$ for general formulas of L_0. ($\alpha', \beta', \cdots$ are composed of propositional variables and $\Box\alpha$, $\Box\beta, \cdots$ with $\sim, \cdot, \vee, \supset$.)

Let γ be an arbitrary formula not containing $\Box$. We call a γ-valuation a manner of value assignments which satisfies the following condition:

$$\text{truth value of } \Box\alpha = \begin{cases} \text{true, if } \gamma\supset\alpha \text{ is a tautology;} \\ \text{false, otherwise,} \end{cases}$$

where α is an arbitrary formula not containing $\Box$, and γ is called the *axiom*.

For a general formula, the following definition is given:

α' is a γ-tautology, if and only if, for a fixed axiom γ, α' is true for all γ-valuation. (α, which does not contain $\Box$, is a γ-tautology if and only if α is a tautology.)

We now state the following theorems:

Theorem 1. *If α' is provable in L_0, then α' is a γ-tautology for all γ.*

Theorem 2. *If α' is a γ-tautology for all γ, then α' is provable in L_0.*

§ 3. Proof of Theorem 1. If $\gamma\supset(\alpha\supset\beta)$ is a tautology and $\gamma\supset\alpha$ is a tautology, then $\gamma\supset\beta$ is a tautology. Therefore $\Box(\alpha\supset\beta)\supset(\Box\alpha\supset\Box\beta)$ is a γ-tautology. If α is a tautology, then $\gamma\supset\alpha$ is true for an arbitrary γ-valuation. Therefore, if α is a tautology, then $\Box\alpha$ is a γ-tautology.

§ 4. Proof of Theorem 2. Let us mention the following lemmata for the sake of the proof of Theorem 2:

Lemma 1. *If $\alpha'\cdot\beta'$ is γ-tautology, then α', β' are γ-tautologies.*

The proof is evident.

Lemma 2. *If δ does not contain $\Box$ and*

(I) $\sim\Box\alpha_1\vee\sim\Box\alpha_2\vee\cdots\vee\sim\Box\alpha_m\vee\Box\beta_1\vee\Box\beta_2\vee\cdots\vee\Box\beta_n\vee\delta$

is a γ-tautology, then

(II) $\sim\Box\alpha_1\vee\sim\Box\alpha_2\vee\cdots\vee\sim\Box\alpha_m\vee\Box\beta_1\vee\Box\beta_2\vee\cdots\vee\Box\beta_n$

is a γ-tautology or δ is a tautology.

Proof. If δ is not a tautology, then there exists a γ-valuation by which δ is false. For such a γ-valuation, (II) is true. Therefore, (II) is a γ-tautology, because truth value of (II) depends only on $\alpha_1, \alpha_2, \cdots, \alpha_m, \beta_1, \beta_2, \cdots, \beta_n, \gamma$ and is independent of γ-valuation, q.e.d.

An arbitrary fomula α' is reduced to a conjunction of the formulas of the form (I). If δ in (I) is a tautology then (I) is provable in L_0. From Lemma 1 and Lemma 2, therefore, for the proof of Theorem 2, it is sufficient to consider (II) as α'. Now take $\alpha_1\cdot\alpha_2\cdot\cdots\cdot\alpha_m$ as axiom γ (if $m=0$ then it means a tautology). Then, since

$$\Box\alpha_1, \Box\alpha_2, \cdots, \Box\alpha_m$$

are true,

$$\Box\beta_1 \vee \Box\beta_2 \vee \cdots \vee \Box\beta_n$$

is true. Therefore for some $i\,(1 \leqq i \leqq n)\,\Box\beta_i$ is true, that is

$$(\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_m) \supset \beta_i$$

is a tautology, accordingly so is

$$\alpha_1 \supset (\alpha_2 \supset (\cdots (\alpha_m \supset \beta_i) \cdots)).$$

Therefore,

$$\Box\alpha_1 \supset (\Box\alpha_2 \supset (\cdots (\Box\alpha_m \supset \Box\beta_i) \cdots))$$

is provable in L_0, and

$$\sim\Box\alpha_1 \vee \sim\Box\alpha_2 \vee \cdots \vee \sim\Box\alpha_m \vee \Box\beta_1 \vee \Box\beta_2 \vee \cdots \vee \Box\beta_n$$

is provable in L_0.

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References

[1] S. Saito: Circular definitions and analyticity. Inquiry (Oslo), **5**, 158–162 (1962).
[2] R. Carnap: Meaning and Necessity. University of Chicago Press, Chicago (1947).