71. On Holomorphic Markov Processes

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Under appropriate regularity conditions, a temporally homogeneous Markov process is associated with a contraction semi-group $\{T_t; t\geqq 0\}$ of class (C_0) [1] in a suitable Banach space X. In certain cases where X are complex Banach spaces, T_t admits a holomorphic extension T_λ given by strongly convergent Taylor series for all $x\in X$:

$$(1)\qquad T_\lambda x=\sum_{n=0}^{\infty}\frac{(\lambda-t)^n}{n!}T_t^{(n)}x \text{ for } \frac{|\lambda-t|}{t}\leqq \text{some positive constant } C,$$

the existence of the n-th strong derivative $T_t^{(n)}x$ in x of $T_t x$ being assumed for any $t>0$ and any $x\in X$ $(n=1, 2, \cdots)$. Such is the case of the semi-group

$$(2)\qquad \begin{aligned}(T_t f)(x)&=(2\pi t)^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{-|x-y|^2/2t}f(y)dy \qquad &(t>0),\\ &=f(x) &(t=0)\end{aligned}$$

in the Banach space $C[-\infty, \infty]$ of bounded uniformly continuous, complex valued functions $f(x)$ on $(-\infty, \infty)$ endowed with the maximum norm. Suggested by this example, we shall call a Markov process a *holomorphic Markov process* if the associated semi-group T_t admits a holomorphic extension T_t of the form given in (1). This notion seems to be of some interest. For instance, we can prove

Proposition. *Let a semi-group T_t with the infinitesimal generator A be associated with a holomorphic Markov process through*

$$(3)\qquad (T_t f)(x)=\int P(t, x, dy)f(y), \qquad f\in X$$

where $P(t, x, dy)$ is the transition probability of this process. Suppose that $T_{t_0}f_0=0$ for some $t_0>0$ and $f_0\in X$. Then $f_0=0$.

Proof. We have $A^n T_{t_0}f_0=T_{t_0}^{(n)}f_0=0$ $(n=0, 1, \cdots)$ by the linearity of A. Hence, by Taylor expansion (1), we see that $T_\lambda f_0=0$ for $|\lambda-t|/t\leqq C$. Repeating the argument, we easily see that $T_t f_0=0$ for all $t>0$ and so $f_0=s-\lim_{t\downarrow 0} T_t f_0=0$.

There are abundant examples of holomorphic Markov processes. In fact, the fractional power [2] $\hat{A}_\alpha$ $(0<\alpha<1)$ of the infinitesimal generator A of a contraction semi-group T_t of class (C_0) generates a constraction semi-group $\hat{T}_{t,\alpha}$ of class (C_0) which admits a holomorphic extension $\hat{T}_{\lambda,\alpha}$ of the similar form given in (1). Moreover, since

(4) $\hat{T}_{t,\alpha}x=\int_0^\infty f_{t,\alpha}(s)T_s x ds$ with a function $f_{t,\alpha}(s)\geqq 0$ satisfying $\int_0^\infty f_{t,\alpha}(s)ds=1$,

we see that $\hat{T}_{t,\alpha}$ is associated with a holomorphic Markov process if T_t is associated with a Markov process.

The purpose of the present note is to devise another method for the construction of holomorphic Markov processes. It is based upon

Theorem. *Let B be the infinitesimal generator of an equi-continuous group of class (C_0) in a complex Banach space X. Then $A=B^2$ is the infinitesimal generator of an equi-continuous semi-group of class (C_0) which is also a holomorphic semi-group* [3] *characterized by any one of the following three conditions:*

(I) *For all $t>0$, $T_tX\subseteqq D(A)$, the domain of A, and there exists a positive constant C_1 such that the family of operators $\{(C_1 t T_t')^n; 0<t\leqq 1, n=0, 1, \cdots\}$ is equi-continuous.*

(II) *T_t admits a holomorphic extension T_λ of the form given in* (1) *such that the family of operators $\{e^{-\lambda}T_\lambda; |\arg\lambda|\leqq\tan(k^{-1}C_1)$ with some fixed $k>0\}$ is equi-continuous.*

(III) *There exists a positive constant C_2 such that the family of operators $\{(C_2\lambda(\lambda I-A)^{-1})^n; Re(\lambda)\geqq 1$ and $n=0, 1, \cdots\}$ is equi-continuous.*

Proof. Since B generates an equi-continuous group of class (C_0), $D(B)$ is dense in X and the resolvents $(\sqrt{\lambda}I\pm B)^{-1}$ both exist as bounded linear operators on X into X for $Re(\sqrt{\lambda})>0$ satisfying the condition

(5) $\{(Re(\sqrt{\lambda})(\sqrt{\lambda}I\pm B)^{-1})^n; Re(\sqrt{\lambda})>0$ and $n=0, 1, \cdots\}$
is equi-continuous.

Thus, by

(6) $$(\lambda I-A)^{-1}=(\sqrt{\lambda}I-B)^{-1}(\sqrt{\lambda}I+B)^{-1}(Re(\lambda)>0)$$

we see that $D(A)=$the range of $(\lambda I-A)^{-1}$ is dense in X with $D(B)$. (6) implies also that

(7) $$\{(\lambda(\lambda I-A)^{-1})^n =(\sqrt{\lambda}(\sqrt{\lambda}I-B)^{-1}\cdot\sqrt{\lambda}(\sqrt{\lambda}I+B)^{-1})^n; \lambda>0, n=0, 1, 2, \cdots\}$$
is equi-continuous.

Hence A generates an equi-continuous semi-group of class (C_0). Moreover,

$$\left\{\left(\left(\sqrt{1+\tau^2}\cos^2\left(\frac{1}{2}\tan^{-1}\tau\right)((1+i\tau)I-A)^{-1}\right)\right)^n\right\}$$
$$=\{(Re(\sqrt{1+i\tau})(\sqrt{1+i\tau}I-B)^{-1}\,Re(\sqrt{1+i\tau})(\sqrt{1+i\tau}\,I+B)^{-1})^n\}$$

is equi-continuous in $-\infty<\tau<\infty$ and in $n=0, 1, \cdots$. Hence, by (III), the operator A generates a holomorphic semi-group.

An example of holomorphic Markov processes. Let $X=C[-\infty, \infty]$ and consider the operator

$$A=a^2(x)\frac{d^2}{dx^2}+b(x)\frac{d}{dx}+q(x). \tag{8}$$

Suppose that $a(x), a'(x), b(x)$, and $q(x)$ are uniformly continuous, bounded real-valued functions in $(-\infty, \infty)$ satisfying conditions

(9) $q(x)\leqq 0$ and $0<\delta\leqq a(x)$ in $(-\infty, \infty)$, where δ is a positive constant.

Then A generates a contraction holomorphic semi-group T_t in X which is *positive*, i.e., $f(x)\geqq 0$ in $(-\infty, \infty)$ implies $(T_tf)(x)\geqq 0$ in $(-\infty, \infty)$. Thus T_t is associated with a holomorphic Markov process.

Proof. A may be written as

(8)′ $A=\left(a(x)\frac{d}{dx}\right)^2+p(x)\frac{d}{dx}-\varepsilon\frac{d}{dx}+q(x)$, where

$$p(x)=b(x)-a(x)a'(x)+\varepsilon \text{ and } \varepsilon>2\sup_{-\infty<x<\infty}|[b(x)-a(x)a'(x)]|.$$

We shall prove (i): $E=\left(a\frac{d}{dx}\right)^2$ generates a contraction positive holomorphic semi-group in X, (ii): $p\frac{d}{dx}$ and $-\varepsilon\frac{d}{dx}$ both generate contraction positive semi-groups of class (C_0) in X, (iii): $q(x)$ generates a contraction positive semi-group of class (C_0) in X, (iv): for $1>\alpha>\frac{1}{2}$, the domain $D\left(p\frac{d}{dx}\right)$ contains the domain $D(\hat{E}_\alpha)$, where $\hat{E}_\alpha$ is the fractional power operator of E and (v): for $1>\alpha>\frac{1}{2}$, the domain $D\left(-\varepsilon\frac{d}{dx}\right)$ contains the domain $D(\hat{F}_\alpha)$, where $\hat{F}_\alpha$ is the fractional power operator of $F=E+p\frac{d}{dx}$ with the domain $D(F)=D(E)$.

Then, by a theorem proved in a preceding note in these Proceedings [4], $F=\left(E+p\frac{d}{dx}\right)$ generates a contraction holomorphic semi-group in X. By H. F. Trotter's product formula [5], we have

$$e^{t\left(E+p\frac{d}{dx}\right)}=s-\lim_{n\to\infty}\left(e^{\frac{t}{n}E}\cdot e^{\frac{t}{n}p\frac{d}{dx}}\right)^n$$

so that the semi-group $e^{t\left(E+p\frac{d}{dx}\right)}$ generated by F is positive by the positivity of semi-groups e^{tE} and $e^{tp\frac{d}{dx}}$. Similarly, by (v), $F-\varepsilon\frac{d}{dx}$ with the domain $D\left(F-\varepsilon\frac{d}{dx}\right)=D(F)$ generates a positive contraction holomorphic semi-group in X. The multiplication operator q is a bounded operator in X which generates a positive contraction semi-group of class (C_0) in X by (iii). Hence, by a similar argument as above, $A=F-\varepsilon\frac{d}{dx}+q$ with the domain $D(A)=D(F)$ generates a positive contraction holomorphic semi-group in X.

The proof of (i) through (iv) is given as follows.

(i): $B=a\frac{d}{dx}$ generates a positive contraction group of class (C_0) in X of translations

$$f(x(y))\rightarrow f(x(y\pm t)),\ \text{where}\ y(x)=\int_0^x a(s)^{-1}ds.$$

Hence the resolvents $(\sqrt{\lambda}\,I\pm B)^{-1}$ are positive operators in X for $\lambda>0$ and so $(\lambda I-E)^{-1}=(\sqrt{\lambda}\,I-B)^{-1}(\sqrt{\lambda}\,I+B)^{-1}$ is a positive operator in X. Thus, remembering the Theorem and the representation $e^{tE}=s-\lim_{n\to\infty}\left(I-\frac{t}{n}E\right)^{-n}$, we have proved (i).

(ii): As in (i), we prove that $p\frac{d}{dx}$ and $-\varepsilon\frac{d}{dx}$ both generate positive contraction group of class (C_0) in X.

(iv): The resolvent of $\hat{E}_\alpha$ is given by T. Kato's formula [6]

$$(\lambda I-\hat{E}_\alpha)^{-1}=\frac{\sin\alpha\pi}{\pi}\int_0^\infty(rI-E)^{-1}\frac{r^\alpha}{\lambda^2-2\lambda r^\alpha\cos\alpha\pi+r^{2\alpha}}dr. \tag{10}$$

We have

$$\begin{aligned}B(rI-E)^{-1}&=B(\sqrt{r}\,I-B)^{-1}(\sqrt{r}\,I+B)^{-1}\\&=(\sqrt{r}\,(\sqrt{r}\,I-B)^{-1}-I)(\sqrt{r}\,I+B)^{-1}\end{aligned}$$

and so, by $\|(\sqrt{r}\,I+B)^{-1}\|\leqq r^{-1/2}$, we see that the right side of

$$B(\lambda I-\hat{E}_\alpha)^{-1}=\frac{\sin\alpha\pi}{\pi}\int_0^\infty B(rI-E)^{-1}\frac{r^\alpha}{\lambda^2-2\lambda r^\alpha\cos\alpha\pi+r^{2\alpha}}dr$$

converges when $1>\alpha>\frac{1}{2}$. This proves (iv).

(v): Remembering

$$\begin{aligned}B(rI-F)^{-1}&=B(I-F)^{-1}(I-F)(rI-F)^{-1}\\&=B(I-F)\cdot[(rI-F)^{-1}+I-r(rI-F)^{-1}]\\&=O(r^{-1})\qquad\text{for small } r\end{aligned}$$

and

$$\begin{aligned}B(rI-F)^{-1}&=B(rI-E)^{-1}\left\{I-p\frac{d}{dx}(rI-E)^{-1}\right\}^{-1}\\&=O(r^{-1/2})\qquad\text{for large } r,\end{aligned}$$

we prove (v) as in (iv).

Remark. That $b(x)$ in (8) may change sign on $(-\infty,\infty)$ was suggested, thanks to a conversation with Professor S. Ito and Dr. H. Tanaka.

References

[1] E. Hille-R. S. Phillips: Functional Analysis and Semi-groups. Providence (1957); K. Yosida: Functional Analysis. Springer (1965).
[2] K. Yosida: Loc. cit.
[3] ——: Loc. cit.
[4] ——: A perturbation theorem for semi-groups of linear operators. Proc. Japan Acad., **41** (8), 645-647 (1965).
[5] H. F. Trotter: On the product of semi-groups of operators. Proc. Amer. Math. Soc., **10**, 545-551 (1959).
[6] See, for instance, K. Yosida: Loc. cit.