96. A Remark on a Comprehension Axiom without Negation

By Takashi NAGASHIMA

Department of the Foundations of Mathematical Sciences, Tokyo University of Education

(Comm. by Zyoiti SUETUNA, M.J.A., May 12, 1966)

In his paper [2], Skolem has proved the consistency of the system of the following axioms

(1) $$\exists y \forall z (z \in y \vdash\dashv \mathfrak{F}(z))$$

and

(2) $$\forall x \forall y (\forall z (z \in x \vdash\dashv z \in y) \vdash (\mathfrak{G}(x) \vdash\dashv \mathfrak{G}(y))),$$

where the formula $\mathfrak{F}$ in (1) is constructed only from $\in$, $\curlyvee$, $\curlywedge$, $\bigwedge$, $\bigvee$ and variables. The purpose of this paper is to show the following generalization of this Skolem's theorem. This generalization implies Namba's result in [1] as well.

Theorem. *The system of axioms* (1) *and* (2) *is consistent if no logical constants except* $\curlyvee$, $\curlywedge$, $\bigwedge$, $\bigvee$, $\forall$, *and* $\exists$ *occur in* $\mathfrak{F}$.

For the proof of his theorem, Skolem has introduced a model satisfying the axioms. Now it will be shown that Skolem's model satisfies the axioms even if $\mathfrak{F}$ has quantifiers.

Let $\mathfrak{D}$ be a domain of individuals A_i, $B_i (0 \leq i < \omega)$ and W, and we define the relation $\in$ as follows:

(i) $A_i \in B_j$ if and only if $j \leq i$,

(ii) $B_i \in A_j$ if and only if $i < j$,

(iii) $A_i \notin A_j$ and $B_i \in B_j$ for all i and j,

(iv) $A_i \notin W$, $W \notin A_i$, $B_i \in W$, and $W \in B_i$ for all i,

(v) $W \in W$.

Then $\mathfrak{D}$ is linearly ordered as

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq W \subseteq \cdots \subseteq B_1 \subseteq B_0,$$

where $\subseteq$ denotes the inclusion relation as usual. Therefore the operations $a \cap b$ and $a \cup b$ can be defined for any a and b in $\mathfrak{D}$. The individuals A_0 and B_0 are written O and V respectively.

We define the operations $*$ and E as follows:

$$A_i^* = A_{i+1}, \quad W^* = W, \quad B_i^* = B_{i+1},$$

$$E(a, b) = \begin{cases} V \text{ if } a \in b, \\ O \text{ otherwise.} \end{cases}$$

Then $\forall x (x \in W \vdash\dashv x \in x)$, $\forall x (x \in a^* \vdash\dashv a \in x)$, and $\forall x (x \in E(a, b) \vdash\dashv a \in b)$ are valid in $\mathfrak{D}$. Since the operations $\cap$, $\cup$, $*$, and E are monotone with respect to $\subseteq$, we have the following

Proposition 1. *If $f(a)$ is constructed from O, V, W, $\cap$, $\cup$, $*$, E, and variables, then*

$$\forall x \forall y (x \subseteq y \vdash f(x) \subseteq f(y))$$

is valid in $\mathfrak{D}$.

Consequently, we obtain the following

Proposition 2. *Let $f(a)$ be a term consisting of O, V, W, $\cap$, $\cup$, $*$, E, and variables, such that*

$$\forall z (z \in f(a) \vdash\dashv \mathfrak{H}(z, a))$$

is satisfied in $\mathfrak{D}$. Then

$$\forall z (z \in f(O) \vdash\dashv \forall x \mathfrak{H}(z, x))$$

and

$$\forall z (z \in f(V) \vdash\dashv \exists x \mathfrak{H}(z, x))$$

are satisfied in $\mathfrak{D}$.

Proposition 3. *For any formula $\mathfrak{F}$ containing no other logical constants than $\curlyvee$, $\curlywedge$, $\bigwedge$, $\bigvee$, $\forall$, and $\exists$, there exists a term t constructed only from O, V, W, $\cap$, $\cup$, $*$, E, and variables, such that*

$$\forall z (z \in t \vdash\dashv \mathfrak{F}(z))$$

is satisfied in $\mathfrak{D}$.

This proposition can be easily proved by induction on the construction of the formula $\mathfrak{F}$. Since axiom (2) is evidently valid in $\mathfrak{D}$, this completes the proof of Theorem.

Let $\mathfrak{D}'$ be the model whose only difference from $\mathfrak{D}$ is that $\in$ is defined by (i)–(iv) and $W \notin W$. Then axioms (1) and (2) are satisfied in $\mathfrak{D}'$ as well. As shown in [2], there is no finite model satisfying axiom (1).

References

[1] K. Namba: On a comprehension axiom without negation. Ann. Japan Assoc. Philos. Sci., **2**, 258-271 (1965).

[2] Th. Skolem: Investigations on a comprehension axiom without negation in the defining propositional functions. Notre Dame J. Formal Logic, **1**, 13-22 (1960).