133. *Operators of Discrete Analytic Functions*

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1. The convolution product. We are concerned with complex-valued functions $f(x, y)$ of two independent integral variables x and y satisfying the following condition.

Let x and y be any integers, and put

$$f_0=f(x, y),\quad f_1=f(x+1, y),\ f_2=f(x+1, y+1),\ f_3=f(x, y+1),$$
$$\bar{f}_0=(f_0+f_1)/2,\ \bar{f}_1=(f_1+f_2)/2,\ \bar{f}_2=(f_2+f_3)/2,\quad \bar{f}_3=(f_3+f_0)/2.$$

Let $p(\neq 1)$ is an arbitrary real or complex number, then

$$\bar{f}_2-\bar{f}_0=p(\bar{f}_1-\bar{f}_3)$$

is equivalent to

$$L_q f\equiv f_0+qf_1-f_2-qf_3=0, \tag{1.1}$$

where $q=(1+p)/(1-p)$.

The function f is said to be *discrete analytic* in R, if the condition (1.1) is satisfied for every x and y in a simply connected region R in the x-y plane. The set of all discrete analytic functions in R is denoted by $\boldsymbol{A}(R)$, or briefly $\boldsymbol{A}$. Duffin's discrete analytic functions [1], [2] are the special case whence $p=q=i$.

Denote for brevity

$$f(x, y)\equiv f(z),\ z\equiv(x, y),\ z_r\equiv(x_r, y_r)$$

where x_r and y_r are integers. The points of the x-y plane with integer coordinates are called *lattice points*.

Let z_r, z_{r+1} be consecutive lattice points. The *double dot integral* along a chain $z_0, \cdots, z_r, z_{r+1}, \cdots, z_n$ is defined by

$$\int_{z_0}^{z_n} f(t)\colon g(t)\delta t\equiv\sum_{r=0}^{n-1}\bar{f}_r\bar{g}_r\delta_r,\quad \delta_r=\pm 1,\ \pm p, \tag{1.2}$$

where $\bar{f}_r=[f(z_r)+f(z_{r+1})]/2$, $\bar{g}_r=[g(z_r)+g(z_{r+1})]/2$,
$\delta_r=1$ or -1 respectively if $y_{r+1}=y_r$ and $x_{r+1}=x_r+1$ or $x_{r+1}=x_r-1$, and $\delta_r=p$ or $-p$ respectively if $x_{r+1}=x_r$ and $y_{r+1}=y_r+1$ or $y_{r+1}=y_r-1$.

The double dot integral of two integral variables

$$\int_0^z f(z-t)\colon g(t)\delta t$$

is said the *convolution product* of $f(x, y)$ and $g(x, y)$, and is denoted by $f*g$, i.e.

$$(f*g)(z)\equiv\int_0^z f(z-t)\colon g(t)\delta t, \tag{1.3}$$

where $0=(0, 0)$ and $z=(x, y)$.

Equation (1.3) requires that not only the chain $0=z_0, z_1, \cdots, z_n=z$ lies in R, but also the chain $z-z_0$, $z-z_1\cdots$, $z-z_n$ lies in R.

Then we have following theorems similar to those in [1], [2].

Theorem 1.1. *If f and $g \in \boldsymbol{A}(R)$, the convolution product* (1.3) *is independent of the path of integration in R, and the operation $*$ is commutative, i.e.*

$$f * g = g * f. \tag{1.4}$$

Further the convolution product $(f * g)(z)$ is discrete analytic in R.

Theorem 1.2. *If f, g, and $h \in \boldsymbol{A}(R)$ in a rectangular region R containing the origin, then the operation $*$ is associative, i.e.*

$$(f * g) * h = f * (g * h). \tag{1.5}$$

We can uniquely determine the values of $f(x, y)$ in a finite rectangular region R by the condition (1.1) for the values of f at lattice points on the x and y axes in R. If $f \in \boldsymbol{A}(R)$ and $f \notin \boldsymbol{A}(E-R)$, we can extend f so that $f \in \boldsymbol{A}(E)$, $R \subset E$, defining suitably the values of f in $E-R$. Thus we have the region of analyticity of the finite rectangular domain or the whole x-y plane. We can restrict (x, y) to be in the first quadrant of the x-y plane without losing the generality.

Let z_{n-1}, z_n be consecutive lattice points. If

$$\bar{f}_{z_n} = [f(z_{n-1}) + f(z_n)]/2 = 0 \quad \text{for all} \quad n = 1, 2, 3, \cdots,$$

then $f(z)$ is called *pseudo zero function* and is denoted by $f(z) = 0^*$, and let us denote the class of all pseudo zero functions by $\boldsymbol{A}_0$. Therefore if $f \in \boldsymbol{A}_0$, then

$$f(z_n) = \begin{cases} c, & \text{for even } n \\ -c, & \text{for odd } n. \end{cases}$$

We define hereafter the mean of $f(x, y)$ on the axes as follows:

$$\bar{f}_{m,0} = [f(m, 0) + f(m-1, 0)]/2$$
$$\bar{f}_{0,n} = [f(0, n) + f(0, n-1)]/2.$$

The class $\boldsymbol{A}(R)$ of discrete analytic functions is classified into the following three classes $\boldsymbol{A}_0$, $\boldsymbol{A}_1$, and $\boldsymbol{A}_2$.

1) $\boldsymbol{A}_0$ is the class of functions of $\boldsymbol{A}(R)$ such that $\bar{f}_{n,0} = 0$ and $\bar{f}_{0,n} = 0$ for all n.

2) $\boldsymbol{A}_1$ consists of two classes $\boldsymbol{A}_x$ and $\boldsymbol{A}_y$. $\boldsymbol{A}_x$ is the class of functions of $\boldsymbol{A}(R)$ such that

$$\bar{f}_{m,0} = 0 \text{ for all } m \quad \text{and} \quad \bar{f}_{0,n} \neq 0 \text{ for some } n.$$

$\boldsymbol{A}_y$ is the class of functions of $\boldsymbol{A}(R)$ such that

$$\bar{f}_{0,n} = 0 \text{ for all } n \quad \text{and} \quad \bar{f}_{m,0} \neq 0 \text{ for some } m.$$

3) $\boldsymbol{A}_2$ is the class of functions of $\boldsymbol{A}(R)$ such that $\bar{f}_{m,0} \neq 0$ and $\bar{f}_{0,n} \neq 0$ for some m, n.

We obtain the following table on the convolution product $f * g$.

Since the convolution product $f * g$ is independent of the path of integration, when f and $g \in \boldsymbol{A}$, we will take hereafter the path $[(0, 0) \to (m, 0) \to (m, n)]$ or $[(0, 0) \to (0, n) \to (m, n)]$. From the Table I we have the following theorem and corollary.

Theorem 1.3. *Suppose that $f*g\equiv 0$, $f, g\in \boldsymbol{A}$. If $g\in \boldsymbol{A}_2$, then $f\in \boldsymbol{A}_0$.*

Corollary. *Suppose that f_1, $f_2\in \boldsymbol{A}$ and $g\in \boldsymbol{A}_2$, then $f_1*g=f_2*g$ implies $f_1=f_2+0^*$.*

Table I.

g \ f		A_0	A_1: A_x	A_1: A_y	A_2
A_0		0	0		0
A_1	A_x	0	A_x	0	A_x
	A_y		0	A_y	A_y
A_2		0	A_x	A_y	A_2

2. Convolution quotient and Operator.

Theorem 2.1. *Suppose that $f*g=h$, f, g, and $h\in \boldsymbol{A}$.*

If $h(0,0)=0$, $\bar{g}_{1,0}\neq 0$, and $\bar{g}_{0,1}\neq 0$, then the function $f(x,y)$ is uniquely determined by the given functions g and h for an initial condition $f(0,0)=c$.

Corollary. *When*

$$\begin{cases}\bar{g}_{1,0}=\bar{g}_{2,0}=\cdots=\bar{g}_{m-1,0}=0,\ \bar{g}_{m,0}\neq 0,\\ \bar{g}_{0,1}=\bar{g}_{0,2}=\cdots=\bar{g}_{0,n-1}=0,\ \bar{g}_{0,n}\neq 0,\end{cases}\tag{2.1}$$

the following condition (2.2) is the necessary and sufficient condition that $f\in \boldsymbol{A}$ is uniquely determined from $f*g=h$ $(g, h\in \boldsymbol{A})$ for $f(0,0)=c$.

$$\begin{cases}h(0,0)=h(1,0)=h(2,0)=\cdots=h(m-1,0)=0, \quad \text{and}\\ h(0,1)=h(0,2)=\cdots=h(0,n-1)=0.\end{cases}\tag{2.2}$$

When $f*g=h$, where $g\in \boldsymbol{A}_2$, $h\in \boldsymbol{A}$, we denote that

$$f=h/g.\tag{2.3}$$

If h does not satisfy (2.2) then $f\notin \boldsymbol{A}$ and $f\in \boldsymbol{Op}$, where $\boldsymbol{Op}$ is a set of operators, the definition of which will be given soon.

Consider the set $\boldsymbol{A}$ of all discrete analytic functions $f(x,y)$ defined at every lattice point in the first quadrant. Then the set $\boldsymbol{A}$ is a *commutative ring* with respect to usual addition and convolutional multiplication.

We consider now *ordered pairs* (a, b) of elements a, b of $\boldsymbol{A}$, where $b\in \boldsymbol{A}_2$. Two ordered pairs (a,b) and (c,d) are said to be *equivalent* if and only if $a*d=b*c$, and the equivalence relation is denoted by

$$(a,b)\equiv(c,d).\tag{2.4}$$

It is proved that the relation $\equiv$ satisfies the usual *equivalence relation.* A class of pairs which are equivalent to an ordered pair (a,b), $b\in \boldsymbol{A}_2$, is called an *operator*, and is denoted by a/b. In order that the set of operators contains the set of functions of $\boldsymbol{A}$, we identify a function $a\in \boldsymbol{A}$ with the following operator:

$$a=(a*k)/k \quad (k\in \boldsymbol{A}_2).\tag{2.5}$$

It is easy to see that (2.5) does not depend on the choice of k. Thus we see $\boldsymbol{Op}\supset \boldsymbol{A}$, where $\boldsymbol{Op}$ denotes the set of operators.

Addition and multiplication in $\boldsymbol{Op}$ are defined as follows.

$$
(2.6)\qquad \begin{cases} \dfrac{a}{b}+\dfrac{c}{d}=\dfrac{a*d+b*c}{b*d}, \\ \dfrac{a}{b}\cdot\dfrac{c}{d}=\dfrac{a*c}{b*d} \quad (b,\, d\in \boldsymbol{A}_2). \end{cases}
$$

Then the set $\boldsymbol{Op}$ is a *commutative ring* with respect to addition and multiplication.

Example 1. *Numerical operator* $[\alpha]$. The operator $(\alpha a)/a$, $(a\in \boldsymbol{A}_2)$ is called the *numerical operator*, and is denoted by $[\alpha]$ or α for brevity, where α is a real or complex number.

Example 2. *Integral operator* l. A function f such that $f(x, y)=1$ is an element of $\boldsymbol{A}$, and is expressed by (2.5) as follows:

$$
(2.7)\qquad 1=(1*f)/f=\left(\int_0^z f\delta t\right)/f \quad (f\in \boldsymbol{A}_2).
$$

Hence $f(x, y)=1$ corresponds to an *integral operator* and is denoted by l as an operator.

Example 3. *Derivative operator* s. The convolutional inverse of the operator l is called the *derivative operator* and is denoted by s.

$$
(2.8)\qquad s=[1]/l=f\Big/\left(\int_0^z f\delta t\right) \quad (f\in \boldsymbol{A}_2).
$$

3. Pseudo power and pseudo fractional power. R. J. Duffin discussed in [1] the *n-th pseudo power* $z^{(n)}$, which is defined by

$$
(3.1)\qquad z^{(n)}=n\int_0^z t^{(n-1)}\delta t, \quad z^{(0)}=1,
$$

and he proved $z^{(n)}\in \boldsymbol{A}$. R. J. Duffin and C. S. Duris proved in [2] the following equalities:

$$
(3.2)\qquad n!\int_0^z\int_0^{t_1}\cdots\int_0^{t_n} f(t_{n+1})\delta t_{n+1}\cdots\delta t_1=\int_0^z (z-t)^{(n)}: f(t)\delta t.
$$

$$
(3.3)\qquad \frac{z^{(n)}}{n!}*\frac{z^{(m)}}{m!}=\frac{z^{(n+m+1)}}{(n+m+1)!}.
$$

These are evident from the point of view of operators, since

$$
(3.4)\qquad \frac{z^{(n)}}{n!}=l^{n+1} \quad (n\text{: positive integer}).
$$

Pseudo powers of $f\in \boldsymbol{A}$ are denoted as follows:

$$
(3.5)\qquad \overbrace{f*f*\cdots*f}^{n}=f^{*n}.
$$

Theorem 3.1. *Suppose that* $f\in \boldsymbol{A}$ *and* $f(0, 0)=0$. *Then there exists* $g\in \boldsymbol{A}$ *such that*

$$
(3.6)\qquad g^{*n}=f \quad (n\text{: positive integer})
$$

if $f(1, 0)\neq 0$ *and* $f(0, 1)\neq 0$.

Corollary. *A necessary and sufficient condition that there exist solutions* g *of the equation*

$$
(3.7)\qquad g^{*n}=f \quad (f,\, g\in \boldsymbol{A})
$$

is as follows:

$$(3.8)\quad \begin{cases} f(0,0)=f(1,0)=\cdots=f(pn,0)=0, & f(pn+1,0)\neq 0, \quad \text{and} \\ f(0,1)=f(0,2)=\cdots=f(0,qn)=0, & f(0,qn+1)\neq 0 \end{cases}$$

$$\begin{pmatrix} p=0,1,2,\cdots \\ q=0,1,2,\cdots \end{pmatrix}.$$

If the condition (3.8) does not hold, the solutions of (3.7) may or may not exist in $\boldsymbol{Op}$. Namely, we have

Theorem 3.2. *Suppose that*

$$(3.9)\quad \begin{cases} \bar{f}_{1,0}=\bar{f}_{2,0}=\cdots=\bar{f}_{p-1,0}=0, & \bar{f}_{p,0}\neq 0, \quad \textit{and} \\ \bar{f}_{0,1}=\bar{f}_{0,2}=\cdots=\bar{f}_{0,q-1}=0, & \bar{f}_{0,q}\neq 0. \quad \textit{Then} \end{cases}$$

(1) *there exists* $x \in \boldsymbol{Op}$ *such that* $x^{*n}=f$, *if* $p\equiv 1$ *(mod n) and* $q\equiv 1$ *(mod n), and*

(2) *there is not exist* $x \in \boldsymbol{Op}$ *such that* $x^{*n}=f$, *if* $p\not\equiv 1$ *(mod n) or* $q\not\equiv 1$ *(mod n).*

We denote hereafter one of *pseudo n-th roots g of* $f\in \boldsymbol{A}$, such that $g(0,0)=0$, by

$$(3.10)\qquad g=f^{*\frac{1}{n}}.$$

Then general one of pseudo n-th roots of $f\in \boldsymbol{A}$, such that $g_1(0,0)=c$, is given by $g_1=f^{*\frac{1}{n}}+0^*$.

For example we define that

$$\frac{z^{\left(\frac{m}{n}\right)}}{\Gamma\left(\frac{m}{n}+1\right)}=\left\{\frac{z^{(n+m-1)}}{\Gamma(n+m)}\right\}^{*\frac{1}{n}},$$

to which corresponds operationally

$$l^{\frac{m}{n}+1}=(l^{n+m})^{\frac{1}{n}}.$$

The detailed proofs of the results obtained in this paper will be published in [3].

References

[1] R. J. Duffin: Basic properties of discrete analytic functions. Duke Math. Jour., **23**, 335-363 (1956).

[2] R. J. Duffin and C. S. Duris: A convolution product for discrete function theory. Duke Math. Jour., **31**, 199-220 (1964).

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[4] J. Mikusiński: Operational Calculus. Warszawa (1959).