168. On Characterizations of I-Algebra. II

By Shôtarô TANAKA

(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 12, 1966)

In this paper, we shall prove that an axiom system of implicational calculus given by Wajsberg is equivalent to Tarski-Bernays' axiom system using an algebraic formulation.

In his paper [2], by the algebraic technique Prof. K. Iséki has proved that Tarski-Bernays' axiom system implies Wajsberg's system and other systems. Further Prof. K. Iséki refers Tarski-Bernays' axiom system as I-algebra. We shall show that Wajsberg axiom implies Tarski-Bernays system. We shall carry out the proof algebraically.

Let $\langle X, 0, *\rangle$ be an abstract algebra. (For the notion of this algebra and notations, see [1].) The algebraic formulations of Wajsberg axioms are given as the following 1)-2), D1-D3.

1) $x\leq x^*(y^*x)$,

2) $(x^*y)^*(z^*u)\leq x^*(z^*(u^*y))$,

D1 $x\leq y$ means $x^*y=0$,

D2 $0\leq x$,

D3 $x\leq y$, $y\leq x$ imply $x=y$.

In 2) we put $z=x$, $y=x$, then we have $(x^*x)^*(x^*u)\leq x^*(x^*(u^*x))$. The right side is equal to 0 by putting $y=u$ in 1). Hence by D1, D2, and D3, we have

3) $x^*x\leq x^*u$.

Let $u=x^*(y^*x)$ in 3), then by 1) we have

4) $x\leq x$.

In 2) putting $x=(z^*x)^*(z^*y)$, $y=y^*x$, and $u=z$, then we have $(((z^*x)^*(z^*y))^*(y^*x))^*(z^*z)\leq((z^*x)^*(z^*y))^*(z^*(z^*(y^*x)))$. The right side is equal to 0 by putting $x=z$, $y=x$, $u=y$ in 2). Further the second term of the left side equal to 0 by 4). Hence we have

5) $(z^*x)^*(z^*y)\leq y^*x$.

If we substitute $z^*(u^*y)$ for x in 2), then the right side is equal to 0 by 4). Hence we have

6) $(z^*(u^*y))^*y\leq(z^*u)$.

Putting $u=y$, $y=x$, and $z=y$ in 6), then by 4) we have

7) $y^*(y^*x)\leq x$.

If we put $y=y^*(y^*x)$ in 5), then by 7) we have

8) $z^*x\leq z^*(y^*(y^*x))$.

Let $x=(z^*y)^*x$, $y=(z^*x)^*(z^*(z^*y))$, $z=(z^*x)^*y$ in 5), then we have

$$(((z^*x)^*y)^*((z^*y)^*x))^*(((z^*x)^*y)^*((z^*x)^*(z^*(z^*y))))$$
$$\leq((z^*x)^*(z^*(z^*y)))^*((z^*y)^*x).$$

The right side is equal to 0 by substituting z^*y for y in 5). Hence we have

9) $((z^*x)^*y)^*((z^*y)^*x)\leq((z^*x)^*y)^*((z^*x)^*(z^*(z^*y)))$.

In 7), put $y=y^*(y^*x)$, then by 7) we have

10) $z^*x\leq z^*(y^*(y^*x))$.

The right side of 9) is the same form with the above formula 10), if we put $x=y$, $y=z$, and $z=z^*x$ in 10). Therefore we have

11) $(z^*x)^*y\leq(z^*y)^*x$.

If we put $z=y$ in 11), then we have $(y^*x)^*y\leq(y^*y)^*x$. On the other hand by D2 and 4), we have $0^*x=0$ and $y^*y=0$. Hence we have

12) $y^*x\leq y$.

Formulas 1), 5), and 12) are axioms of Tarski-Bernays' system. Therefore the proof is complete.

References

[1] K. Iséki: Algebraic formulations of propositional calculi. Proc. Japan Acad., **41**, 803-807 (1965).

[2] Y. Imai and K. Iéski: On axiom systems of propositional Calculi. XIV. Proc. Japan Acad., **42**, 19-22 (1966).