157. On J-Groups of Spaces which are Like Projective Planes

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(Comm. by Zyoiti SUETUNA, M.J.A., Sept. 12, 1966)

Let K be a CW-complex obtained from attaching a $2n$-cell V^{2n} to the n-sphere S^n by a map $f: S^{2n-1}\to S^n$. We call K a space which is like real, complex, quaternian, Cayley projective plane in accordance with $n=1, 2, 4, 8$. Our purpose is to calculate J-groups of $K^{(*)}$. Since J-group of a space is determined by its homotopy type we shall use the following notations:

$$P_R(m)=S^1\cup_f e^2, \quad (f)\in \pi_1(S^1)=Z[\iota], \quad (f)=m[\iota]$$
$$P_C(m)=S^2\cup_f e^4, \quad (f)\in \pi_3(S^2)=Z[h], \quad (f)=m[h]$$
$$P_Q(m, n)=S^4\cup_f e^8, \quad (f)\in \pi_7(S^4)=Z[\nu]+Z_{12}[\tau], \quad (f)=m[\nu]+n[\tau]$$
$$P_K(m, n)=S^8\cup_f e^{16}, \quad (f)\in \pi_{15}(S^8)=Z[\sigma]+Z_{120}[\rho], \quad (f)=m[\sigma]+n[\rho]$$

where $[\iota]$, $[h]$, $[\nu]$, $[\tau]$, $[\sigma]$, $[\rho]$ are the generators of respective homotopy groups and $[\iota_4, \iota_4]=2[h]+[\tau]$, $[\iota_8, \iota_8]=2[\sigma]+\rho$.

For example $P_R(2)$, $P_C(1)$, $P_Q(1, 0)$, $P_K(1, 0)$ have respectively the same homotopy type as real, complex, quaternion, Cayley projective planes. Now let $\widetilde{KO}(X)$ denote the abelian group formed by all stable real vecter bundles over X. Then there exists the natural onto-homomorphism $J: \widetilde{KO}(X)\to J(X)$ by the definition of $J(X)$. Hence in order to determine $J(X)$ it is sufficient to calculate $\widetilde{KO}(X)$ and the kernel of J.

1. Case of $P_R(m)$. If m is odd we have $\widetilde{KO}(P_R(m))$ is trivial and therefore $J(P_R(m))$ is also trivial. If m is even we have $J^{-1}(0)=0$ by the following

Lemma 1. *The commutative diagram is exact:*

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \widetilde{KO}(S^2) \arrow[r, "p^*"'] \arrow[d] & \widetilde{KO}(P_R(m)) \arrow[r, "i^*"'] \arrow[d] & \widetilde{KO}(S^1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J(S^2) \arrow[r, "p^*"'] \arrow[d] & J(P_R(m)) \arrow[r, "i^*"'] \arrow[d] & J(S^1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(*) J. F. Adames: On the group $J(X)$-1, Topology, Vol. 2 (1963).

where p denotes the map: $S^n \smile e^{2n} \to S^{2n}$ which pinchs S^n to a point. On the other hand we can obtain

Lemma 2. $\widetilde{KO}(P_R(m)) = Z_2 + Z_2 \quad$ *if* $m \equiv 0 \quad$ *mod* 4,
$= Z_4 \quad$ *if* $m \equiv 2 \quad$ *mod* 4.

The proof of Lemmas 1 and 2 is easy. Thus we have

Proposition 1. $J(P_R(m)) = 0 \quad$ *if* $m \equiv 1 \quad$ *mod* 2,
$= Z_2 + Z_2 \quad$ *if* $m \equiv 0 \quad$ *mod* 4,
$= Z_4 \quad$ *if* $m \equiv 2 \quad$ *mod* 4.

2. Case of $P_C(m)$. By calculation of the kernel of p^* we have

Lemma 3. *The following sequences are exact*:

$$0 \to J(S^4) \to J(P_C((m)) \to J(S^2) \to 0 \qquad \text{if } m \equiv 0 \quad \text{mod } 2,$$
$$0 \to 2J(S^4) \to J(P_C(m)) \to J(S^2) \to 0 \qquad \text{if } m \equiv 1 \quad \text{mod } 2.$$

Since it is easy to obtain

Lemma 4. $\widetilde{KO}(P_C(m)) = Z + Z_2 \quad$ *if* $m \equiv 0 \quad$ *mod* 2,
$= Z \quad$ *if* $m \equiv 1 \quad$ *mod* 2.

We have therefore

Proposition 2. $J(P_C(m)) = Z_2 + Z_{24} \quad$ *if* $m \equiv 0 \quad$ *mod* 2,
$= Z_{24} \quad$ *if* $m \equiv 1 \quad$ *mod* 2.

3. Case of $P_Q(m, n)$ and $P_K(m, n)$. We start from

Lemma 5. *The following sequence is exact*:

$$0 \to J(S^{2l}) \to J(X) \to J(S^l) \to 0,$$

where X denotes $P_Q(m, n)$ or $P_K(m, n)$ in accordance with $l=4$ or 8.

In the case of $l=4$ this lemma is easy but it seems to need some device in the case of $l=8$ in proving the exactness of a partial sequence $J(S^{16}) \to J(X) \to J(S^8)$. Now let K be a CW-complex obtained from attaching a 4l-cell V^{4l} to a CW-complex L and let P be the map: $K \to S^{4l}$ which pinches L to a point. Suppose that

1) $J(S^{4l}) \xrightarrow[P^*]{} J(K) \xrightarrow[i^*]{} J(L)$ is exact where i is the inclusion map: $L \to K$;

2) the order of $J(S^{4l})$ is the denominater of the rational number $B_l/4l$ expressed as a fraction in lowest term;

3) L is torsion free.

Then we have

Lemma 6. *for* $\xi \in \widetilde{KO}(K)$, $J(\xi)=0$ *if and only if* $i^*(\xi)=0$ *and* $\hat{A}(\xi) = 1 + ch_2(x) + ch_4(x) + \cdots$ *for some* $x \in \widetilde{KO}(K)$ *where* $\hat{A}$ *denotes* $\hat{A}$*-genus and* $ch_{2k}(x)$ *denotes* k*-th Chern character of* x.

On the other hand it is trivial that a stable real vecter bundle over $P_Q(m, n)$ and $P_K(m, n)$ is determined by Pontrjagin classes. With respect to theses Pontrjagin classes we can have:

Lemma 7. *For* $\xi \in \widetilde{KO}(P_Q(m, n))$ *there exist two integers* a, b *such that*

$$P_1(\xi)=2ae^4, \qquad P_2(\xi)=(a(2a-1)m+2na+6b)e^8,$$

and also converse is true.

Lemma 8. *For* $\eta\in\widetilde{KO}(P_K(m,n))$ *there exist two integers* c, d *such that*

$$P_2(\eta)=6ce^8, \qquad P_4(\eta)=(18c(c-1)m+39cm-42cn+7!\,d)e^{16}.$$

By combining with Lemmas 6, 7, and 8 we have

Proposition 3. *Let* (A, B) *denotes the greatest common divisor of* A *and* B. *Then*

$$J(P_Q(m,n))=Z_{1440/(m-2n,60)}+Z_{4(m-2n,60)},$$

$$J(P_K(m,n))=Z_{115200/(m-2n,\,80)}+Z_{(m-2n,\,80)}.$$

Let $P_R(2)$, $P_C(2)$, $P_Q(2)$, $P_K(2)$ denote real, complex, quaternion, Cayley projective planes respectively. Then we have the

Corollary. $J(P_R(2))=Z_4$, $J(P_C(2))=Z_{24}$, $J(P_Q(2))=Z_{1140}+Z_4$,
$J(P_K(2))=Z_{115200}$.