198. *On an Integral Inequality of the Stepanoff Type and its Applications*

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§ 1. *Generalized Hilbert transforms.* Let $f(x)$ be a measurable function which is defined on the real line. Then the Hilbert transform is defined by the following formula:

$$\tilde{f}(x)=P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}dt=\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-t|>\varepsilon}\frac{f(t)}{x-t}dt, \tag{1.1}$$

(cf. E. C. Titchmarsh [6 Chap. V]).
But as for $f(x)$ to be bounded, the Hilbert transform does not necessarily exist. In that place, by $W_p(p\geqq 1)$ let us denote the class of measurable functions such that

$$\int_{-\infty}^{\infty}\frac{|f(x)|^p}{1+x^2}dx<\infty. \tag{1.2}$$

Then for a function in W_p, we can define a generalized Hilbert transform:

$$\tilde{f}^*(x)=P.V.\frac{x+i}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\,\frac{dt}{x-t}. \tag{1.3}$$

From the identity:

$$\frac{x-i}{(t+i)(x-t)}=\frac{(x-t)+(t+i)}{(t+i)(x-t)}=\frac{1}{t+i}+\frac{1}{x-t}$$

we have formally

$$\begin{aligned}\tilde{f}^*(x)&=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}dt+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}dt\\&=\tilde{f}(x)+A_f.\end{aligned} \tag{1.4}$$

This modified definition is due to H. Kober [3] and N. I. Achiezer [1]. In our previous paper [4] we have studied this operator systematically applying N. Wiener's generalized harmonic analysis [7]. Let us introduce another class of functions. By $S_p(p\geqq 1)$ we shall denote the class of functions satisfying the following condition.

$$\sup_{-\infty<x<\infty}\left(\frac{1}{l}\int_{x}^{x+l}|f(t)|^p dt\right)^{\frac{1}{p}}<\infty. \tag{1.5}$$

Then it is plain that $S_p\subset W_p$. Now we shall prove the following inequality.

Theorem 1. *Let $f(x)$ belong to the class $S_p(p>1)$. Then we have for any positive number $l>0$,*

$$\sup_{-\infty<x<\infty}\left(\frac{1}{l}\int_x^{x+l}|\tilde{f}^*(t)|^p dt\right)^{\frac{1}{p}} \leqq A_p \sup_{-\infty<x<\infty}\left(\frac{1}{l}\int_x^{x+l}|f(t)|^p dt\right)^{\frac{1}{p}} \tag{1.6}$$

where $A_p=O(1/p-1)$ *as* $p\to 1$.

§ 2. *Proof of the Theorem* 1. Let us denote by I and I' two intervals $(u-\pi, u+\pi)$ and $(u-2\pi, u+2\pi)$ respectively. Let us also introduce conjugate function on the interval I'

$$\tilde{f}_{I'}(x)=P.V.\frac{(-1)}{\pi}\int_{I'} f(t)\frac{1}{4}\cot\frac{1}{4}(t-x)dt.$$

Then we can write

$$\tilde{f}^*(x)-\tilde{f}_{I'}(x)=\frac{1}{\pi}\int_{I'}\frac{f(t)}{t+i}dt+\frac{1}{\pi}\int_{I'}f(t)\left\{\frac{1}{x-t}-\frac{(-1)}{4}\cot\frac{1}{4}(t-x)\right\}dt$$

$$+\frac{x+i}{\pi}\int_{(-\infty,\infty)-I'}\frac{f(t)}{t+i}\,\frac{dt}{x-t}=J_1+J_2+J_3,\ \text{say}.$$

As for J_1 we have by the Hölder inequality,

$$\int_{u-\pi}^{u+\pi}|J_1|^p dx \leqq \frac{1}{\pi^p}\int_{u-\pi}^{u+\pi}dx\int_{I'}|f(t)|^p dt\left(\int_{I'}\frac{dt}{|t+i|^q}\right)^{\frac{p}{q}}$$

$$\leqq A_1\int_{I'}|f(t)|^p dt.$$

Next as for J_2, the property of the kernel

$$K(x,t)=\frac{1}{x-t}-\frac{(-1)}{4}\cot\frac{1}{4}(t-x)=O(t-x)$$

providing $x\in I$ and $t\in I'$, reads the following estimation

$$\int_{u-\pi}^{u+\pi}|J_2|^p dx \leqq \frac{A_2}{\pi^p}\int_{u-\pi}^{u+\pi}dx\int_{I'}|f(t)|^p dt\left(\int_{I'}|t-x|^q dt\right)^{\frac{p}{q}}$$

$$=A_1'\int_{I'}|f(t)|^p dt.$$

In the last as for J_3, decomposing the integral into small pieces, we have

$$\frac{1}{\pi}\int_{u+2\pi}^{\infty}f(t)\frac{x+i}{(t+i)(x-t)}dt$$

$$\leqq\sum_{j=1}^{\infty}\frac{1}{\pi}\left(\int_{u+2j\pi}^{u+2(j+1)\pi}|f(t)|^p dt\right)^{\frac{1}{p}}\left(\int_{u+2j\pi}^{u+2(j+1)\pi}\left|\frac{x+i}{(t+i)(x-t)}\right|^q dt\right)^{\frac{1}{q}}$$

$$\leqq\frac{B_1}{\pi}\sup_{-\infty<u<\infty}\left(\int_{u-\pi}^{u+\pi}|f(t)|^p dt\right)^{\frac{1}{p}}\left\{\left(\int_{u+2\pi}^{\infty}\frac{dt}{|t+i|^q}\right)^{\frac{1}{q}}+\left(\int_{u+2\pi}^{\infty}\frac{dt}{|x-t|^q}\right)^{\frac{1}{q}}\right\}$$

$$\leqq B_1'\sup_{-\infty<u<\infty}\left(\int_{u-\pi}^{u+\pi}|f(t)|^p dt\right)^{\frac{1}{p}},$$

where x runs over the interval I. Therefore we obtain

$$\int_{u-\pi}^{u+\pi}\left|\frac{1}{\pi}\int_{u+2\pi}^{\infty}f(t)\frac{x+i}{(t+i)(x-t)}dt\right|^p dx \leqq B_1'\sup_{-\infty<u<\infty}\int_{u-\pi}^{u+\pi}|f(t)|^p dt.$$

Similarly it is obtained that

$$\int_{u-\pi}^{u+\pi}\left|\frac{1}{\pi}\int_{-\infty}^{u-2\pi} f(t)\frac{x+i}{(t+i)(x-t)}dt\right|^p dx \leqq B_2' \sup_{-\infty<u<\infty}\int_{u-\pi}^{u+\pi}|f(t)|^p dt.$$

Thus we have

$$\int_{u-\pi}^{u+\pi}|J_3|^p dx \leqq B_3 \sup_{-\infty<x<\infty}\int_{u-\pi}^{u+\pi}|f(t)|^p dt.$$

Applying the M. Riesz theorem we [8] have

$$\int_{u-\pi}^{u+\pi}|f_{I'}(x)|^p dx \leqq A_p^p \int_{u-2\pi}^{u+2\pi}|f(t)|^p dt, \quad (p>1)$$

where $A_p = O(1/p-1)$ as $p \to 1$. Combining these estimations we have proved the theorem.

§ 3. *Almost periodicity of Hilbert transforms.* In our previous paper [5] we studied the almost periodicity in the sense of Stepanoff (cf. A. S. Besicovitch [2]). Now we have it without any additional condition.

Theorem 2. *Let $f(x)$ be S_p-almost periodic* ($p>1$). Then the generalized Hilbert transform $\tilde{f}^*(x)$ also does. Furthermore if we denote the Fourier series of f as follows

(3.1) $$f(x) \sim \sum c_n e^{i\lambda_n x}$$

then those of $\tilde{f}^*(x)$ are

(3.2) $$\tilde{f}^*(x) \sim \tilde{c}_0 + \sum{}'(-i \operatorname{sign} \lambda_n) c_n e^{i\lambda_n x}$$

where the prime means that the term $\lambda_n = 0$ is omitted from the summation and

(3.3) $$\tilde{c}_0 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \tilde{f}^*(x)dx.$$

Proof. By a simple calculation of complex variable method, for any trigonometrical polynomial $p(x) = \sum a_n e^{i\lambda_n x}$ we have $\tilde{p}^*(x) = A_p + \sum{}'(-i \operatorname{sign} \lambda_n) a_n e^{i\lambda_n x}$, where $A_p = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{p(t)}{t+i}dt$. The existence of the A_p is guaranteed by the well known theorem (cf. E.C. Titchmarsh [6, Chap. I, § 1.10]). Then the remaining part is obvious by the theorem 1 and the well known theorem of almost periodic functions (cf. A. S. Besicovitch [2, Chap. II, § 8]).

Now we shall quote the Bochner theorem:

Theorem (S. Bochner). Let $f(x)$ be S_1-almost periodic and uniformly continuous. Then $f(x)$ is uniformly almost periodic.

Combining theorem 2 and that of S. Bochner we have the following theorem immediately

Theorem 3. *Let $f(x)$ be uniformly almost periodic and its generalized Hilbert transform $\tilde{f}^*(x)$ be uniformly continuous.* Then the $\tilde{f}^*(x)$ is uniformly almost periodic.

§ 4. *The Lipschitz condition of the Hilbert transform.* In this section we shall prove

Theorem 4. *Let us assume that $f(x)$ belong to the class*

$W_p(p\geqq 1)$ *and satisfy the Lipschitz condition of order* $\alpha, 0<\alpha<1$ or $\alpha=1$. Then we have

(i) $\quad |\tilde{f}^*(x)-\tilde{f}^*(y)|=0(|x-y|^{\alpha})$, if $0<\alpha<1$

and

(ii) $\quad |\tilde{f}^*(x)-\tilde{f}^*(y)|=0(|x-y|\log|x-y|^{-1})$, if $\alpha=1$

respectively

Proof. Let us suppose that x and y belong to the interval $I=(u-\pi, u+\pi)\subset(u-2\pi, u+2\pi)=I'$. Then by the formula (1.4) we obtain

$$\tilde{f}^*(x)-\tilde{f}^*(y)=\int_{(-\infty,\infty)-I'} f(t)(1-\chi_I)\frac{-(x-y)}{(x-t)(y-t)}dt$$
$$+\int_{I'} f(t)\chi_I\Big(\frac{1}{x-t}-\frac{1}{y-t}\Big)dt=J_1+J_2, \text{ say,}$$

where $\chi_I(t)=1$ if $t\in I$; $=0$, if $t\notin I'$; $0<\chi_I(t)<1$, if $t\in I'-I$; and continuously differentiable function. As for J_1 we have

$$|J_1|\leqq A_1(|x-y|)\Big(\int_{-\infty}^{\infty}\frac{|f(t)|^p}{1+t^2}dt\Big)^{\frac{1}{p}}\Big(\int_{-\infty}^{\infty}\frac{dt}{1+t^2}\Big)^{\frac{1}{q}}=A_1'|x-y|$$

if $p>1$ and

$$|J_1|\leqq A_2(|x-y|)\int_{-\infty}^{\infty}\frac{|f(t)|}{1+t^2}dt=A_2'|x-y|$$

if $p=1$ respectively. As for J_2 we shall quote the Titchmarsh theorem [6, Chap. V, §5.15], then we have $J_2=B_1(|x=y|^{\alpha})$, if $0<\alpha<1$; $=(|x-y|\log|x-y|^{-1})$ if $\alpha=1$ respectively. Besides, constants A_1', A_2', B_1, and B_2 are independent on u, thus we have proved the theorem. Combining theorems 2, 3, and 4 we obtain a corollay.

Corollay 1. *Let $f(x)$ be uniformly almost periodic and satisfy the Lipschitz condition of order α, $0<\alpha<1$. Then the generalized Hilbert transform $\tilde{f}^*(x)$ is uniformly almost periodic.*

§5. *Concluding remarks.* From the formal equality (1.4), the existence of the ordinary Hilbert transform $\tilde{f}(x)$ and that of A_f are equivalent to each other. In the case of $f(x)$ to be S_1-almost periodic, one of the sufficient condition is as follows: there exist a positive number δ such that $|\lambda_m-\lambda_n|>\delta$ for any pair of m and n. In this case the almost periodicity of the generalized Hilbert transform reads that of the ordinary one. The case $p=1$ is still open.

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