195. On the Convergence of Semi-Groups of Operators

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1. Let X be a locally convex, sequentially complete, linear topological space and $\{U_t^{(n)} : t \geqq 0\}_{n=1,2,\dots}$ be a sequence of semi-groups of operators on X, satisfying the following conditions:

(i) $U_0^{(n)} = I,\ U_t^{(n)} U_{t'}^{(n)} = U_{t+t'}^{(n)}, \qquad t, t' \geqq 0,$

(ii) $\lim_{t \to t_0} U_t^{(n)} x = U_{t_0}^{(n)} x, \qquad t_0 \geqq 0,\ x \in X,$[1]

(iii) $\{U_t^{(n)}\}$ are equi-continuous in t and n, i.e., for any continuous semi-norm p on X, there exists a continuous semi-norm q on X, independent of t and n, such that

$$p(U_t^{(n)} x) \leqq q(x) \qquad x \in X.$$

And let $F^{(n)}$ be the infinitesimal generator of $\{U_t^{(n)}\}_{t \geqq 0}$ i.e.,

$$F^{(n)} x = \lim_{h \to 0} h^{-1}(U_h^{(n)} - I)x.$$

We consider the following condition (A):

(A) *There exists a dense linear subset* $\mathfrak{M} \subset \bigcup_{n \geqq 1} \bigcap_{k \geqq n} \mathcal{D}(F^{(k)})$ *such that*

$$\lim_{n,n'} (F^{(n)} x - F^{(n')} x) = 0 \qquad \textit{for each } x \in \mathfrak{M}.$$

M. Hasegawa [2] considered the following problem in the case of Banach space: Under the condition (A), is it true that the additive operator $F = \lim_n F^{(n)}$ or some closed extension of F is the infinitesimal generator of a semi-group $\{U_t\}$ which satisfies $U_t = \lim_n U_t^{(n)}$?

In this paper we shall extend Hasegawa's Theorem on the space X mentioned above and obtain the main theorem:

Theorem 3. *We assume the condition* (A) *and put*

$$Fx = \lim_n F^{(n)} x, \qquad X \in \mathfrak{M}.$$

Then there exists a closed extension $\tilde{F}$ of the F and it generates an equi-continuous semi-group $\{U_t\}$ of class (C_0), where

$$U_t x = \lim U_t^{(n)} x, \qquad \textit{for all } x \in X \textit{ and } t \geqq 0,$$

if and only if the following condition (H) *is satisfied:*

(H) *For some $\lambda_0 > 0$ and for any continuous semi-norm p on X,*

$$\lim_{n,n'} p((I - \lambda_0^{-1} F^{(n)})^{-1} x - (I - \lambda_0^{-1} F^{(n')})^{-1} x) = 0, \qquad x \in X.$$

The proof is given in the section 3.

On the other hand, T. Kato [1] has obtained the following

1) A Semi-group satisfying the conditions (i) and (ii) is said to be of class (C_0).

theorem.

Theorem 1. *Under the condition,*

(K) *for some* $\lambda_0>0$, $\lim_n R(\lambda_0; F^{(n)})x=I(\lambda_0)x$ *exists for all* x *of* X *and* $\overline{R(I(\lambda_0))}=X$,

the limit $\lim_n R(\lambda; F^{(n)})x=I(\lambda)x$ *exists for each* $\lambda>0$ *and* $x\in X$, *and* $I(\lambda)$ *is the resolvent of the infinitesimal generator* $\tilde{F}$ *of an equi-continuous semi-group* $\{U_t\}$ *of class* (C_0).

Furthermore,

$$U_t x=\lim_n U_t^{(n)}x \qquad \text{for every } x\in X,$$

where the limit holds uniformly on every compact interval of $t\geqq 0$.

Theorem 1 doesn't give any informations about the relations among $F^{(n)}$ and $\tilde{F}$. We shall make them clear by Theorem 3. And we shall give some conditions which are equivalent to (H) under the condition (A) in the section 2.

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2. We begin with defining the operators $I_\lambda^{(n)}$ from X into itself, by

$$I_\lambda^{(n)}x=\lambda R(\lambda; F^{(n)})x=(I-\lambda^{-1}F^{(n)})^{-1}x, \qquad x\in X.$$

Theorem 2. *Under* (A), *the conditions* (H), (K) *and the following conditions* (H_1), (H_2), (H_3), *and* (H_4) *are mutually equivalent.*

(H_1) *For any* $t, t'\geqq 0$ *and any continuous semi-norm* p *on* X,

$$\lim_{n,n'} p(U_t^{(n)}U_{t'}^{(n')}x-U_{t'}^{(n')}U_t^{(n)}x)=0, \qquad x\in X.$$

(H_2) *For any* $\lambda, \lambda'>0$ *and any continuous semi-norm* p *on* X,

$$\lim_{n,n'} p(I_\lambda^{(n)}I_{\lambda'}^{(n')}x-I_{\lambda'}^{(n')}I_\lambda^{(n)}x)=0, \qquad x\in X.$$

(H_3) *For any* $t\geqq 0$ *and any continuous semi-norm* p *on* X,

$$\lim_{n,n'} p(U_t^{(n)}x-U_t^{(n')}x)=0, \qquad x\in X.$$

(H_4) *For any* $\lambda>0$ *and any continuous semi-norm* p *on* X,

$$\lim_{n,n'} p(I_\lambda^{(n)}x-I_\lambda^{(n')}x)=0, \qquad x\in X.$$

Proof. $(\mathrm{H}_1)\Rightarrow(\mathrm{H}_2)$. Using the relation between $I_\lambda^{(n)}$ and $U_t^{(n)}$ [1; p 240], we have

$$p(I_\lambda^{(n)}I_{\lambda'}^{(n')}x-I_{\lambda'}^{(n')}I_\lambda^{(n)}x)\leqq\lambda\lambda'\int_0^\infty\int_0^\infty e^{-\lambda s}e^{-\lambda' t}p(U_s^{(n)}U_t^{(n')}x-U_t^{(n')}U_s^{(n)}x)dsdt.$$

Now we may use only the Lebesgue convergence theorem.

$(\mathrm{H}_2)\Rightarrow(\mathrm{H}_4)$. The limit $\lim_n F^{(n)}x=Fx$ exists for each $x\in\mathfrak{M}$ since X is sequentially complete. If $x\in\mathfrak{M}$, then there exists an n_0 such that $x\in\bigcap_{n\geqq n_0}\mathcal{D}(F^{(n)})$. Thus by choosing $n, n'\geqq n_0$, we have

$$x=I_\lambda^{(n)}(I-\lambda^{-1}F^{(n)})x=I_\lambda^{(n')}(I-\lambda^{-1}F^{(n')})x.$$

From the equi-continuity of $I_\lambda^{(n)}$ with respect to n and $\lambda>0$, for any continuous semi-norm p on X, there exist continuous semi-norms q and q' on X such that

$$
\begin{aligned}
&p(I_\lambda^{(n)}x-I_\lambda^{(n')}x)\\
&\quad=p(I_\lambda^{(n)}I_\lambda^{(n')}(I-\lambda^{-1}F^{(n')})x-I_\lambda^{(n')}I_\lambda^{(n)}(I-\lambda^{-1}F^{(n)})x)\\
&\quad\leqq p(I_\lambda^{(n)}I_\lambda^{(n')}x-I_\lambda^{(n')}I_\lambda^{(n)}x)+\lambda^{-1}p(I_\lambda^{(n')}I_\lambda^{(n)}(F^{(n)}x-Fx))\\
&\qquad+\lambda^{-1}p(I_\lambda^{(n')}I_\lambda^{(n)}Fx-I_\lambda^{(n)}I_\lambda^{(n')}Fx)+\lambda^{-1}p(I_\lambda^{(n)}I_\lambda^{(n')}(Fx-F^{(n')}x))\\
&\quad\leqq p(I_\lambda^{(n)}I_\lambda^{(n')}x-I_\lambda^{(n')}I_\lambda^{(n)}x)+\lambda^{-1}q(F^{(n)}x-Fx)\\
&\qquad+\lambda^{-1}p(I_\lambda^{(n')}I_\lambda^{(n)}Fx-I_\lambda^{(n)}I_\lambda^{(n')}Fx)+\lambda^{-1}q'(F^{(n')}x-Fx),\qquad x\in\mathfrak{M}.
\end{aligned}
$$

Since each term of this right side tends to zero as $n, n'\to\infty$, for any $x\in\mathfrak{M}$ and any continuous semi-norm p, $\lim_{n,n'} p(I_\lambda^{(n)}x-I_\lambda^{(n')}x)=0$. Thus (H_4) follows from the denseness of $\mathfrak{M}$.

$(\mathrm{H}_4)\Rightarrow(\mathrm{H})$ is obvious.

$(\mathrm{H})\Rightarrow(\mathrm{K})$. Suppose (H), then $\lim_n I_{\lambda_0}^{(n)}x=I_{\lambda_0}x$ exists for each $x\in X$. And we obtain $I_{\lambda_0}(I-\lambda_0^{-1}F)y=y$ for each $y\in\mathfrak{M}$ from

$$
\begin{aligned}
&p(I_{\lambda_0}(I-\lambda_0^{-1}F)y-y)\\
&\leqq p(I_{\lambda_0}(I-\lambda_0^{-1}F)y-I_{\lambda_0}^{(n)}(I-\lambda_0^{-1}F)y)+p(I_{\lambda_0}^{(n)}(I-\lambda_0^{-1}F)y-I_{\lambda_0}^{(n)}(I-\lambda_0^{-1}F^{(n)})y)\\
&\leqq p(I_{\lambda_0}(I-\lambda_0^{-1}F)y-I_{\lambda_0}^{(n)}(I-\lambda_0^{-1}F)y)+\lambda_0^{-1}q(F^{(n)}y-Fy).
\end{aligned}
$$

This shows $\mathcal{R}(I_{\lambda_0})\supset\mathfrak{M}$, and therefore $\mathcal{R}(I_{\lambda_0})$ is dense in X.

$(\mathrm{K})\Rightarrow(\mathrm{H}_3)$ is obvious from Theorem 1.

$(\mathrm{H}_3)\Rightarrow(\mathrm{H}_1)$. $\lim_n U_t^{(n)}x=V_tx$ exists for any $t\geqq 0$ and $x\in X$. Then for any continuous semi-norm p on X, we have

$$
\begin{aligned}
&p(U_t^{(n)}U_{t'}^{(n')}x-U_{t'}^{(n')}U_t^{(n)}x)\\
&\quad\leqq p(U_t^{(n)}U_{t'}^{(n')}x-U_t^{(n)}V_{t'}x)+p(U_t^{(n)}V_{t'}x-U_t^{(n)}U_{t'}^{(n)}x)\\
&\qquad+p(U_{t'}^{(n)}U_t^{(n)}x-U_{t'}^{(n)}V_tx)+p(U_{t'}^{(n)}V_tx-V_{t'}V_tx)\\
&\qquad+p(V_{t'}V_tx-U_{t'}^{(n')}V_tx)+p(U_{t'}^{(n')}V_tx-U_{t'}^{(n')}U_t^{(n)}x)\\
&\quad\leqq q(U_{t'}^{(n')}x-V_{t'}x)+q(V_{t'}x-U_{t'}^{(n)}x)+q(U_t^{(n)}x-V_tx)\\
&\qquad+p(U_{t'}^{(n')}V_tx-V_{t'}V_tx)+p(V_{t'}V_tx-U_{t'}^{(n')}V_tx)+q(V_tx-U_t^{(n)}x).
\end{aligned}
$$

Since each term of the above right hand tends to zero as $n, n'\to\infty$, we get the condition (H_1).

3. We prove Theorem 3 mentioned in the section 1.

Proof of Theorem 3. Since the "only if" part is evident from Theorem 2, we shall prove the "if" part. By virtue of Theorem 2, we assume now (H_4). Setting $I_\lambda x=\lim_n I_\lambda^{(n)}x$ for $\lambda>0$ and $x\in X$, we have $\lim_{\lambda\to\infty}I_\lambda x=x$ for all $x\in X$ and have the resolvent equation

$$
(*)\qquad I_\lambda x=\frac{\lambda'-\lambda}{\lambda'}I_{\lambda'}I_\lambda x+\frac{\lambda}{\lambda'}I_{\lambda'}x\qquad\text{for } x\in X.
$$

In fact, for each continuous semi-norm p and each $x\in X$,

$$
\begin{aligned}
p(I_\lambda x-x)&\leqq p(I_\lambda x-I_\lambda^{(n)}x)+p(I_\lambda^{(n)}x-(I-\lambda^{-1}F^{(n)})I_\lambda^{(n)}x)\\
&\leqq p(I_\lambda x-I_\lambda^{(n)}x)+\lambda^{-1}p(F^{(n)}I_\lambda^{(n)}x)\to 0
\end{aligned}
$$

as $\lambda\to\infty$. For each n and each $x\in X$

$$
I_\lambda^{(n)}=\frac{\lambda'-\lambda}{\lambda'}I_{\lambda'}^{(n)}I_\lambda^{(n)}x+\frac{\lambda}{\lambda'}I_{\lambda'}^{(n)}x.
$$

Passing to the limit as $n\to\infty$, we have resolvent equation $(*)$. The equation (*) shows that $\mathfrak{R}(I_\lambda)$ is independent of $\lambda>0$ and hence we shall denote it as $\mathfrak{R}$. Moreover, it follows that I_λ is a one to one operator between X and $\mathfrak{R}$. In fact, if we assume that for some $\lambda_0>0$, there exists a non-zero element x such that $I_{\lambda_0}x=0$, then from the resolvent equation (*) we have $I_\lambda x=0$ for any $\lambda>0$ and thus $x=\lim\limits_{\lambda\to\infty} I_\lambda x=0$, which is impossible. Thus we can define the operator $\tilde{F}_\lambda=\lambda(I-I_\lambda^{-1})$ on $\mathfrak{R}$, which is independent of λ since

$$\begin{aligned}\tilde{F}_{\lambda_1}I_\lambda x&=\lambda_1(I-I_{\lambda_1}^{-1})I_\lambda x=\lambda_1 I_\lambda x-\lambda_1\left[\frac{\lambda_1-\lambda}{\lambda_1}I_\lambda x+\frac{\lambda}{\lambda_1}x\right]\\&=\lambda(I_\lambda-I)x=\tilde{F}_\lambda I_\lambda x.\end{aligned}$$

Then, similarly as in the proof of Theorem 1, we can prove that the operator $\tilde{F}$ defined by

$$\tilde{F}x=\lambda(I-I_\lambda^{-1})x \qquad \text{for } x\in\mathfrak{R}$$

generates an equi-continuous semi-group $\{U_t\}$ of class (C_0) such that

$$U_t x=\lim_n U_t^{(n)}x \qquad \text{for all } x\in X.$$

On the other hand, similarly as in the proof of Theorem 2, we have $I_\lambda(I-\lambda^{-1}F)y=y$ for each $y\in\mathfrak{M}$, from which it follows that $\mathfrak{M}\subset\mathfrak{R}(I_\lambda)=\mathfrak{R}$. Thus $\tilde{F}$ is a closed extension of the operator F.

Now we consider the following condition (C):

(C) *There exists a dense linear subset* $\mathfrak{M}'\subset X$ *such that* $I_{\lambda_0}\mathfrak{M}'\subset\mathfrak{M}$ *for some* $\lambda_0>0$.

Using this condition, we can extend Trotter's Theorem [3; Th. 5.2] on our space X.

Corollary of Theorem 3. *Under the condition* (A), *the closure* $\bar{F}$ *of* F *is the infinitesimal generator of an equi-continuous semi-group* $\{U_t\}$ *of class* (C_0), *where* $U_t x=\lim\limits_n U_t^{(n)}x$ *for all* $x\in X$ *and* $t\geqq 0$, *if and only if* (H)+(C) *or equivalently the following condition* (T) *is satisfied*:

(T) *There exists a positive real number* λ_0 *such that* $\overline{\mathfrak{R}(I-\lambda_0^{-1}F)}=X$.

Proof. If $\bar{F}$ is the infinitesimal generator, then $\mathfrak{R}(I-\lambda^{-1}\bar{F})=X$ for each $\lambda>0$. The condition (T) follows from the relation of inclusion $\overline{\mathfrak{R}(I-\lambda^{-1}F)}\supset\mathfrak{R}(I-\lambda^{-1}\bar{F})$. Conversely if we put $\mathfrak{M}'=(I-\lambda_0^{-1}F)\mathfrak{M}$, then it can be seen that (T) implies (C). Furthermore we can prove that (T) implies also (H). In fact, from the equi-continuity of $I_{\lambda_0}^{(n)}$ with respect to n, for any continuous semi-norm p on X and $y\in\mathfrak{M}$,

$$\begin{aligned}p(I_{\lambda_0}^{(n)}(I-\lambda_0^{-1}F)y-y)&=p(I_{\lambda_0}^{(n)}(I-\lambda_0^{-1}F)y-I_{\lambda_0}^{(n)}(I-\lambda_0^{-1}F^{(n)})y)\\&\leqq\lambda_0^{-1}q(F^{(n)}y-Fy),\end{aligned}$$

which shows

$$\lim_{n,n'} p(I_{\lambda_0}^{(n)}(I-\lambda_0^{-1}F)y-I_{\lambda_0}^{(n')}(I-\lambda_0^{-1}F)y)=0, \qquad y\in\mathfrak{M}.$$

Thus from $\overline{\mathscr{R}(I-\lambda_0^{-1}F)}=X$, we have

$$\lim_{n\ n'} p(I_{\lambda_0}^{(n)}x - I_{\lambda_0}^{(n')}x) = 0 \qquad \text{for each } x \in X,$$

which is nothing but the condition (H). Next we assume (H)+(C). Then by Theorem 3, there exists a closed extension $\tilde{F}$ which is the infinitesimal generator of $\{U_t\}$. Now it remains to prove $\tilde{F}=\bar{F}$. For any $x \in X$, there exists a generalized sequence $\{x_\alpha\}\subset\mathfrak{M}'$ such that $\lim_\alpha x_\alpha = x$. By virtue of the definition of $\tilde{F}$ in Theorem 3, for any continuous semi-norm p on X, we have

$$\begin{aligned}\lim_\alpha p(\tilde{F}I_{\lambda_0}x - FI_{\lambda_0}x_\alpha) &= \lim_\alpha p(\tilde{F}I_{\lambda_0}(x-x_\alpha)) \\ &= \lim_\alpha p(\lambda_0(I_{\lambda_0}-I)(x-x_\alpha)) = 0,\end{aligned}$$

which shows $\tilde{F}=\bar{F}$.

Remark. By Theorem 2, the condition (H) in Theorem 3 and also in the Corollary can be replaced by any one of the conditions (H_i) and (K).

References

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[3] H. F. Trotter: Approximation of semi-groups of operators. Pac. J. Math., **8**, 887-919 (1958).