162. *On Mappings of Type l^1*

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In this note, we shall improve a result in A. Pietsch [1], and we shall consider the problem 8.4.4 mentioned in [1]. We mainly follow notations in [1].

Let E and F be normed spaces. Let $\mathcal{L}(E, F)$ be all continuous linear mapping of E into F, let $\mathcal{A}(E, F)$ be all mappings $t \in \mathcal{L}(E, F)$ such that $t(E)$ is finite dimensional subspace of F, and let $\mathcal{A}_r(E, F)$ be all mappings $t \in \mathcal{L}(E, F)$ such that $t(E)$ is at most r-dimensional subspace of F.

The norm $\|\ \|$ of t in $\mathcal{L}(E, F)$ is defined by $\|t\| = \sup\{\|t(x)\|; \|x\| \leq 1, x \in E\}$ and the r-th approximation number of t in $\mathcal{L}(E, F)$ is defined by

$$\alpha_r(t) = \inf\{\|t-u\|; u \in \mathcal{A}_r(E, F)\}.$$

By $l^p(E, F)$ we denote all $t \in \mathcal{L}(E, F)$ such that $\sum_{r=0}^{\infty} \alpha_r(t)^p < \infty$ holds for all positive number p, and we put

$$\rho_p(t) = \left\{\sum_{r=0}^{\infty} \alpha_r(t)^p\right\}^{1/p}$$

Lemma. *Each mapping $t \in \mathcal{A}(E, F)$ can be represented in following form*

$$t(x) = \sum_{i=1}^{r} \lambda_i < x, a_i > y_i, \ |\lambda_i| \leqq \|t\| \qquad \textit{for} \quad i = 1, 2, \cdots, r,$$

where $a_i \in U^0$, $y_i \in V$, and λ_i are real or complex numbers.

Proof of this lemma is found in ([1], p. 121).

Proposition. Each mapping $t \in l^p(E, F)$ (where p is any positive number) can be represented in following form, for any positive number δ,

$$t(x) = \sum_{r=0}^{\infty} \lambda_r < x, a_r > y_r$$

and

$$\left\{\sum_{r=0}^{\infty} |\lambda_r|^p\right\}^{1/p} \leqq 2^{1+3/p}(1+\delta)\rho_p(t),$$

where $a_r \in U^0$, $y_r \in V$, and λ_r are real or complex numbers.

Proof. By definition of $\alpha_r(t)$, for all natural number n, there exist u_n in $\mathcal{A}_{2^n-2}(E, F)$ such that $\|t-u_n\| \leqq (1+\delta)\alpha_{2^n-2}(t)$.

Let $v_n = u_{n+1} - u_n$ then we have $d_n \equiv$ dimensional of $v_n(E) \leqq 2^{n+2}$ and

$$\|v_n\| \leqq \|t-u_n\| + \|t-u_{n+1}\| \leqq 2(1+\delta)\alpha_{2^n-2}(t).$$

By lemma, v_n is represented by

$$(1)\qquad v_n(x)=\sum_{i=1}^{dn}\lambda_i^n<x,\ a_i^n>y_i^n,$$

where $a_i^n\in U^0$, $y_i^n\in V$, $|\lambda_i^n|\leqq\|v_n\|$.

Then the sequence $\{\alpha_r(t)\}$ $(r=0, 1, 2, \cdots)$ is a monotone decreasing, therefore

$$\sum_n 2^{n-1}\alpha_{2^n-2}(t)^p\leqq\sum_n\sum_{r=2^{n-1}-1}^{2^n-2}\alpha_r(t)^p=\sum_{r=0}^{\infty}\alpha_r(t)^p=\rho_p(t)^p.$$

On the other hand, $d_n\|v_n\|^p\leqq 2^{p+n+2}(1+\delta)^p\alpha_{2^n-2}(t)^p$ implies

$$\sum_n d_n\|v_n\|^p\leqq 2^{p+3}(1+\delta)^p\rho_p(t)^p.$$

Hence, by (1)

$$\sum_n\sum_{i=1}^{dn}|\lambda_i^n|^p\leqq\sum_n d_n\|v_n\|^p\leqq 2^{p+3}(1+\delta)^p\rho_p(t)^p.$$

Therefore

$$\left\{\sum_n\sum_{i=1}^{dn}|\lambda_i^n|^p\right\}^{1/p}\leqq 2^{1+3/p}(1+\delta)\rho_p(t)$$

and

$$t(x)=\lim_{m\to\infty}u_{m+1}(x)=\sum_n v_n(x)=\sum_n\sum_{i=1}^{dn}\lambda_i^n<x,\ a_i^n>y_i^n.$$

In particular, if p is equal to 1, then we have the following

Theorem. *Any mapping t in $l^1(E, F)$ is a nuclear mapping, and*

$$\nu(t)\leqq 2^4\rho_1(t).$$

By the theorem above, we can replace $\nu(t)\leqq 2^5\rho_1(t)$ into $\nu(t)\leqq 2^4\rho_1(t)$. But we do not know whether 2^4 is the smallest number ρ satisfying $\nu(t)\leqq\rho\rho_1(t)$.

Reference

[1] A. Pietsch: Lokalkonvexe Räume. Berlin (1965).