161. *A Characterization of Lukasiewiczian Algebra. II*

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In [4], I introduced according to algebraic technique used by Prof. K. Iséki [1], the notion of L-algebra and I showed that a L-algebra is a three-valued Lukasiewicz algebra. In this note I shall show that a three-valued Lukasiewicz algebra is a L-algebra, hence the notion of L-algebra is equivalent with the notion of three-valued Lukasiewicz algebra.

A three-valued Lukasiewicz algebra is [3] a system $\langle A, 1, \sim, \mu, \cap, \cup \rangle$ such that the following axioms are verified:

A1) $x \cup 1 = 1$,
A2) $x \cap (x \cup y) = x$,
A3) $x \cap (y \cup z) = (z \cap x) \cup (y \cap x)$,
A4) $x = \sim\sim x$,
A5) $\sim(x \cap y) = \sim x \cup \sim y$,
A6) $\sim x \cup \mu x = 1$,
A7) $x \cap \sim x = \sim x \cap \mu x$,
A8) $\mu(x \cap y) = \mu x \cap \mu y$.

In a three-valued Lukasiewicz algebra hold the followings:

1) $\mu\mu x = \mu x$,
2) $\sim\mu\sim\mu x = \mu x$,
3) $\mu(x \cup y) = \mu x \cup y\mu$,
4) $x \cap \sim x \leq y \cup \sim y$,
5) $x \cap \mu\sim x = x \cap \sim x$,
6) $x \cap \sim\mu x = 0$,[1]
7) $\sim x \cap \sim\mu\sim x = 0$,
8) $\mu x \cap \sim\mu x = 0$,
9) $\mu\sim x \cap \sim\mu\sim x = 0$,
10) $\sim\mu x \cap \sim\mu\sim x = 0$,
11) $\sim\mu\sim x \leq x \leq \mu x$,
12) $x \cap y = 0 \rightleftarrows \mu y \leq \sim\mu x$,
13) $\mu x = \mu y$ and $\sim\mu\sim x = \sim\mu\sim y$ imply $x = y$, which is the Moisil determination principle.

If we note $x * y = (x \cap \sim\mu y) \cup (\sim\mu\sim x \cap \sim y)$, we shall prove that $\langle A, 0, *, \sim \rangle$ is a L-algebra.

Lemma 1. $x * y = 0 \rightleftarrows x \leq y$.

1) We note $0 = \sim 1$.

Really, $x*y=0 \rightleftarrows x\cap\sim\mu y=0$, $\sim\mu\sim x\cap\sim y=0\rightleftarrows\mu x\leq\sim\mu\sim\mu y$, $\mu\sim\mu\sim x\leq\sim\mu\sim y\rightleftarrows\mu x\leq\mu y$, $\sim\mu\sim x\leq\sim\mu\sim y\rightleftarrows x\leq y$.

Lemma 2. $x*y\leq x$.

$(x*y)*x=(((x\cap\sim\mu y)\cup(\sim\mu\sim x\cap\sim y))\cap\sim\mu x)\cup(\sim\mu\sim((x\cap\sim\mu y)$ $\cup(\sim\mu\sim x\cap\sim y))\cap\sim x)=((\sim\mu\sim x\cap\sim\mu y)\cup(\sim\mu\sim x\cap\sim\mu y))\cap\sim x$ $=\sim\mu\sim x\cap\sim\mu y\cap\sim x=0$, hence $x*y\leq x$.

Lemma 3. $x=x*(\sim x*x)$.

We have $\sim x*x=(\sim x\cap\sim\mu x)\cup(\sim\mu\sim\sim x\cap\sim x)=\sim x\cap\sim\mu x=\sim\mu x$, hence $x*(\sim x*x)=x*\sim\mu x=(x\cap\sim\mu\sim\mu x)\cup(\sim\mu\sim x\cap\sim\sim\mu x)=(x\cap\mu x)$ $\cup(\sim\mu\sim x\cap\mu x)=\mu x\cap(x\cup\sim\mu\sim x)=\mu x\cap x=x$.

Lemma 4. $x*\sim y=y*\sim x$.

$x*\sim y=(x\cap\sim\mu\sim y)\cup(\sim\mu\sim x\cap\sim\sim y)=(y\cap\sim\mu\sim x)\cup(\sim\mu\sim y\cap\sim\sim x)$ $=y*\sim x$.

Lemma 5. $\sim x*y=\sim y*x$.

It is easily proved by Lemma 4 for $x=\sim x$, $y=\sim y$.

Lemma 6. $x*(x*\sim x)\leq\sim(y*(y*\sim y))$.

We have $x*\sim x=(x\cap\sim\mu\sim x)\cup(\sim\mu\sim x\cap\sim\sim x)=\sim\mu\sim x$ hence,

$$\begin{aligned}x*(x*\sim x)&=(x\cap\sim\mu\sim\mu\sim x)\cup(\sim\mu\sim x\cap\sim\sim\mu\sim x)\\&=(x\cap\mu\sim x)\cup(\sim\mu\sim x\cap\mu\sim x)=\mu\sim x\cap(x\cup\sim\mu\sim x)\\&=\mu\sim x\cap x=x\cap\sim x;\end{aligned}$$

also $\sim(y*(y*\sim y))=\sim(y\cap\sim y)=y\cup\sim y$ and from $x\cap\sim x\leq y\cup\sim y$ we obtain Lemma 6.

Lemma 7. $(x*y)*(x*z)\leq z*y$.

We have

$$z*y=(z\cap\sim\mu y)\cup(\sim\mu\sim z\cap\sim y)$$

and

$(x*y)*(x*z)=(((x\cap\sim\mu y)\cup(\sim\mu\sim x\cap\sim y))\cap\sim\mu((x\cap\sim\mu z)$ $\cup(\sim\mu\sim x\cap\sim z)))\cup(\sim\mu\sim((x\cap\sim\mu y)\cup(\sim\mu\sim x\cap\sim y))\cap\sim((x\cap\sim\mu z)$ $\cup(\sim\mu\sim x\cap\sim z)))=(((x\cap\sim\mu y)\cup(\sim\mu\sim x\cap\sim y))\cap(\sim\mu x\cup\mu z)$ $\cap(\mu\sim x\cup\sim\mu\sim z))\cup(((\sim\mu\sim x\cap\sim\mu y)\cup(\sim\mu\sim x\cap\sim\mu y))\cap(\sim x\cup\mu z)$ $\cap(\mu\sim x\cup z))=(((x\cap\sim\mu y)\cup(\sim\mu\sim x\cap\sim y))\cap(\sim\mu x\cup\sim\mu\sim z\cup(\mu z$ $\cap\mu\sim x)))\cup(\sim\mu\sim x\cap\sim\mu y\cap(\sim x\cup z\cup(\mu z\cap\mu\sim x)))=(x\cap\sim\mu y$ $\cap\sim\mu\sim z)\cup(x\cap\mu\sim x\cap\sim\mu y\cap\mu z)\cup(\sim\mu\sim x\cap\sim y\cap\sim\mu\sim z)\cup(\sim\mu\sim x$ $\cap\sim\mu y\cap z)$,

hence

$\sim\mu\sim(z*y)=(\sim\mu\sim z\cap\sim\mu y)\cup(\sim\mu\sim z\cap\sim\mu y)=\sim\mu\sim z\cap\sim\mu y$,

$\mu(z*y)=(\mu z\cap\sim\mu y)\cup(\sim\mu\sim z\cap\mu\sim y)$,

$\sim\mu\sim((x*y)*(x*z))=(\sim\mu\sim x\cap\sim\mu y\cap\sim\mu\sim z)\cup(\sim\mu\sim x\cap\mu\sim x\cap$ $\sim\mu y\cap\mu z)\cup(\sim\mu\sim x\cap\sim\mu y\cap\sim\mu\sim z)\cup(\sim\mu\sim x\cap\sim\mu y\cap\sim\mu\sim z)=\sim\mu$ $\sim x\cap\sim\mu y\cap\sim\mu\sim z, \mu((x*y)*(x*z))=(\mu x\cap\sim\mu y\cap\sim\mu\sim z)\cup(\mu x\cap\mu$ $\sim x\cap\sim\mu y\cap\mu z)\cup(\sim\mu\sim x\cap\mu\sim y\cap\sim\mu\sim z)\cup(\sim\mu\sim x\cap\sim\mu y\cap\mu z)$,

whence

$\sim\mu\sim((x*y)*(x*z))\leq\sim\mu\sim(z*y)$,

$\mu((x*y)*(x*z))\leq\mu(z*y)$

and with a consequence of Moisil's determination principle, we have $(x*y)*(x*z)\leq z*y$.

Lemma 8. $x*(x*(z*(z*y)))=z*(z*(y*(y*x)))$.

We have

$z*(z*y)=(z\cap\sim\mu((z\cap\sim\mu y)\cup(\sim\mu\sim z\cap\sim y)))\cup(\sim\mu\sim z\cap\sim((z\cap\sim\mu y)\cup(\sim\mu\sim z\cap\sim y)))=(z\cap\mu y\cap(\mu\sim z\cup\sim\mu\sim y))\cup(\sim\mu\sim z\cap\mu y\cap(\mu\sim z\cup y))=(z\cap\sim z\cap\mu y)\cup(z\cap\sim\mu\sim y)\cup(\sim\mu\sim z\cap y)$
$=(z\cap\mu\sim z\cap\mu y)\cup(z\cap y\cap(\sim\mu\sim y\cup\sim\mu\sim z))$,

hence

$\sim\mu\sim(z*(z*y))=(\sim\mu\sim z\cap\mu\sim z\cap\mu y)\cup(\sim\mu\sim z\cap\sim\mu\sim y\cap(\sim\mu\sim y\cup\sim\mu\sim z))=\sim\mu\sim(z\cap y)$

and

$\mu(z*(z*y))=(\mu z\cap\mu\sim z\cap\mu y)\cup(\mu z\cap\mu y\cap(\sim\mu\sim y\cup\sim\mu\sim z))$
$=\mu z\cap\mu y\cap(\mu\sim z\cup\sim\mu\sim y\cup\sim\mu\sim z)=\mu(z\cap y)$,

whence

$z*(z*y)=z\cap y$.

Then,

$x*(x*(z*(z*y)))=x\cap(z\cap y)=z\cap(y\cap x)=z*(z*(y*(y*x)))$.

Lemma 9. $(x*z)*((x*z)*(y*z))=(y*z)*(y*x)$.

We have

$(x*z)*(y*z)=(((x\cap\sim\mu z)\cup(\sim\mu\sim x\cap\sim z))\cap\sim\mu((y\cap\sim\mu z)\cup(\sim\mu\sim y\cap\sim z)))\cup(\sim\mu\sim((x\cap\sim\mu z)\cup(\sim\mu\sim x\cap\sim z))\cap\sim((y\cap\sim\mu z)\cup(\sim\mu\sim y\cap\sim z)))=(x\cap\sim\mu z\cap\sim\mu y)\cup(\sim\mu\sim x\cap\sim z\cap\sim\mu y)\cup(\sim\mu\sim x\cap\sim z\cap\mu z\cap\mu\sim y)\cup(\sim\mu\sim x\cap\sim\mu z\cap\sim y)$

whence

$(x*z)*((x*z)*(y*z))=(x\cap\sim z\cap(\sim\mu z\cup\sim\mu\sim x)\cap(\sim\mu x\cup\mu z\cup\mu y)\cap(\mu\sim x\cup\sim\mu\sim z\cup\mu y)\cap(\mu\sim x\cup\sim\mu\sim z\cup\sim\mu z\cup\sim\mu\sim y)\cap(\mu\sim x\cup\mu z\cup\sim\mu\sim y))\cup(\sim\mu\sim x\cap\sim\mu z\cap(\sim x\cup\mu z\cup\mu y)\cap(\mu\sim x\cup z\cup\mu y)\cap(\mu\sim x\cup z\cup\sim\mu z\cup\sim\mu\sim y)\cap(\mu\sim x\cup\mu z\cup y)$.

Applying to the preceding relation $\sim\mu\sim$ and μ respectively, we obtain after long calculation

$\sim\mu\sim((x*z)*((x*z)*(y*z)))=\sim\mu\sim x\cap\sim\mu\sim y\cap\sim\mu z$

and

$\mu((x*z)*((x*z)*(y*z)))=(\mu x\cap\mu y\cap\sim\mu z)\cup(\sim\mu\sim x\cap\sim\mu\sim y\cap\mu\sim z)$.

Using the calculation made at the Lemma 7, we have

$\sim\mu\sim((y*z)*(y*x))=\sim\mu\sim y\cap\sim\mu z\cap\sim\mu\sim x$

and

$\mu((y*z)*(y*x))=(\mu y\cap\sim\mu z\cap\sim\mu\sim x)\cup(\mu y\cap\mu\sim y\cap\sim\mu z\cap\mu x)\cup(\sim\mu\sim y\cap\mu\sim z\cap\sim\mu\sim x)\cup(\sim\mu\sim y\cap\sim\mu z\cap\mu x)=(\mu x\cap\sim\mu z\cap((\mu y\cap\mu\sim y)\cup(\sim\mu\sim y)\cup(\sim\mu\sim x\cap\mu y\cap\sim\mu z)\cup(\sim\mu\sim x\cap\sim\mu\sim y\cap\mu\sim z)$

$=(\mu x \cap \mu y \cap \sim \mu z) \cup (\sim \mu \sim x \cap \sim \mu \sim y \cap \mu \sim z)$.

By the preceding relation, we have

$$\sim \mu \sim ((x*z)*((x*z)*(y*z))) = \sim \mu \sim ((y*z)*(y*x)),$$

$$\mu((x*z)*((x*z)*(y*z))) = \mu((y*z)*(x*x))$$

and with the Moisil determination principle, it follows that

$$(x*z)*((x*z)*(y*z)) = (y*z)*(y*x).$$

Hence we have proved that a three-valued Lukasiewicz algebra is a L-algebra. According to the result obtained in [4] we have the following

Theorem. *The notion of L-algebra is equivalent with the notion of three-valued Lukasiewicz algebra.*

References

[1] K. Iséki: A Characterization of Boolean Algebra. Proc. Japan Acad., **41**, 893-897 (1965).

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[4] C. O. Sicoe: A characterization of Lukasiewiczian Algebra. I. Proc. Japan Acad., **43**, 729-732 (1967).