210. *On Some Classes of Operators. II*

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In [2], [4] some classes of non-normal operators were introduced namely the classes $C(N, k)$. The definition of these classes is:

Definition 1. *An operator T on Hilbert space is in $C(N, k)$ if*
$$\| Tx \|^k \leq \| T^k x \|$$
for every unit vector $x \in H$.

The aim of this Note is to give some new results connected with these classes.

Theorem 1. *There exists an operator T which is normaloid and is not in $C(N, k)$ for all k.*

Proof. We consider the operator [6]
$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
and I be the one-dimensional identity operator and put
$$T = A \oplus I.$$

It is very easy to see (this was firstly observed by Toeplitz) that
$$\mathrm{Cl}\ W(A) = \{z, |z| \leqslant 1/2\}$$
(Here $W(A) = \{\langle Ax, x\rangle, \|x\| = 1\}$ and $\mathrm{Cl}\ E$ denotes the closure of such a set E). Also $\mathrm{Cl}\ W(T)$ is the covex hull of $\mathrm{Cl}\ W(A)$ and the point $\{1\}$. Since
$$\sup_{\|x\|=1} |\langle Tx, x\rangle| \geqslant 1 = \|T\|$$
it is clear that T is normaloid.

We consider T as an operator on a finite dimensional space. Then it is clear that if $T \in C(N, k)$, T must be normal by Theorem 3 of [4].

This leads to the fact that T is not in $C(N, k)$ for all k and the theorem is proved.

Remark 1. The construction of examples of operators which are normaloid and are not in $C(N, k)$ for all k has the following reason: the restriction of a normaloid operator to an invariant subspace is not generally normaloid.

The following theorem represents a generalization to our case of results in [10], [11].

Theorem 2. *If $p(\lambda)$ is a polynomial non-vanishing on $\sigma(T) - \{0\}$ and $p(T)$ is a Riesz operator of class $C(N, k)$ for some k then T is normal.*

Proof. The fact that $p(\lambda)$ is non-vanishing on $\sigma(T) - \{0\}$ implies [9] that T is a Riesz-operator. The proof of the fact that T must be normal is modeled on a proof of Theorem 2 of [3]. By Theorem

1 of [4] there exists $\lambda_0 \in \sigma(T)$ such that $|\lambda_0| = ||T||$.

The subspace

$$\eta_T(\lambda_0) = \{x,\ Tx = \lambda_0 x\}$$

is not equal to $\{0\}$ because T is a Riesz operator.

The proof may be continued exactly as the proof of Theorem 2 in [3] and obtain the derived result.

Corollary 1. *If T^m is a Riesz operator in $C(N, k)$ for some $k \geqslant 2$ then every invariant subspace of T reduces.*

Proof. From the above theorem T must be normal and since T^k is a Riesz operator then T is also compact. This implies obviously that T has the desired property.

In connection with this corollary appears the following problem: if T is a Riesz operator and every invariant subspace of T reduces T then necessarily T is normal? We conjecture the affirmative.

Theorem 3. *If T is non-zero Riesz operator of class (N, k) for some integer k then its compact part for any decomposition is a non-zero operator.*

Proof. Suppose that there exists a decomposition in which the compact part is zero. Then T is quasinilpotent. By Theorem 1 of [4] T is zero. (The proof is exactly as for the class $(N, 2)$ in [10])

Theorem 4. *If T is an operator with the following properties*

1. *is hyponormal*
2. *$Tp(T) = C$, C compact*

where $p(\lambda)$ is a polynomial non-vanishing on $\sigma(T) - \{0\}$ then T is normal.

Proof. We consider the Calkin algebra $C = \mathcal{L}(H)/I(H)$ where $\mathcal{L}(H)$ is the Banach algebra of all bounded operators on H, $I(H)$ is the two-sided ideal of compact operators [7]. It is known that C is a B^*-algebra and if T is hyponormal, $\bar{T}$ its image in C is also hyponormal. We have thus

$$\bar{T}p(\bar{T}) = 0.$$

This implies that $\bar{T} = 0$ and thus T is compact. By Ando's theorem T is normal.

As a consequence of Theorem 1 [4] and Theorem 2 of [12] we denote the following:

Theorem 5. *If T is an operator such that $f(T)$ is in some class $C(N, k)$ for all function f with no poles in $\sigma(T)$ then $\sigma(T)$ is a spectral set for T.*

Remark 2. Another example of operator which is convexoid (an operator T is convexoid if Cl $W(T) =$ convex hul of the spectrum) normaloid and not in $C(N, k)$ for all k may be the operator constructed in [5], § 2 (In [5] the class $C\,(N, 2)$ is called the class of paranormal operators).

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