6. *Unions of Strongly Paracompact Spaces. II*[1)]

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As is well known, the space that is the union of two closed strongly paracompact[2)] subspaces need not be strongly paracompact (see [6]). In the previous note (see [8]), we have proved the following theorem:

Theorem 1. *Let $\mathfrak{F}=\{F_\alpha \mid \alpha \in A\}$ be a locally finite closed covering of a regular T_1-space X such that $Fr(F_\alpha)$*[3)] *has the locally Lindelöf property for any $\alpha \in A$. Then a necessary and sufficient condition that X be strongly paracompact is that F_α is strongly paracompact for any $\alpha \in A$.*

A main purpose of this note is to show the following theorem:

Theorem 2. *Let X be a normal T_1-space and $\mathfrak{G}=\{G_\alpha \mid \alpha \in A\}$ be a locally finite open covering of X such that $G_\alpha \cap G_\beta$ has the locally Lindelöf property with respect to its relative topology for each $\alpha, \beta \in A$* with $\alpha \neq \beta$. If *G_α is strongly paracompact for each $\alpha \in A$, then X is strongly paracompact.*

Proof. Suppose that A is well ordered. As is well known ([2]; Proposition 1.2), we can take the open covering $\mathfrak{H}=\{H_\alpha \mid \alpha \in A\}$ of X such that $\bar{H}_\alpha$[4)] $\subset G_\alpha$ for each $\alpha \in A$ and therefore $\bar{\mathfrak{H}}$[5)] is a locally finite closed covering of X. By the way to make the covering $\mathfrak{H}$,

$$X-\Big(\bigcup_{\alpha<\alpha_0} H\Big) \cup \Big(\bigcup_{\alpha>\alpha_0} G_\alpha\Big) \subset H_{\alpha_0} \subset \bar{H}_{\alpha_0} \subset G_{\alpha_0},$$

and then

$$Fr(\bar{H}_{\alpha_0}) \subset \bar{H}_{\alpha_0}-H_{\alpha_0} \subseteq G_{\alpha_0}-\Big[X-\Big\{\Big(\bigcup_{\alpha<\alpha_0} H\Big) \cup \Big(\bigcup_{\alpha>\alpha_0} G\Big)\Big\}\Big]$$
$$\subset G_{\alpha_0} \cap \Big(\bigcup_{\alpha\neq\alpha_0} G_\alpha\Big) = \bigcup_{\alpha\neq\alpha_0} (G_{\alpha_0} \cap G_\alpha),$$

then $\bigcup_{\alpha\neq\alpha_0} (G_{\alpha_0} \cap G_\alpha)$ has the locally Lindelöf property and hence $Fr(\bar{H}_{\alpha_0})$ has the locally Lindelöf property. After all we have the locally finite closed covering $\mathfrak{H}=\{\bar{H}_{\alpha_0} \mid \alpha \in A\}$ such that $Fr(\bar{H}_{\alpha_0})$ has the locally Lindelöf property for any $\alpha \in A$. Therefore, by Theorem 1, we can

1) This note is a continuation of the previous note [8].

2) The Hausdorff space X is *strongly paracompact* if an arbitrary open covering of X has the star finite open covering of X as a refinement.

3) $Fr(F_\alpha)$ denotes the boundary of $\bar{F}_\alpha$ in X, that is, $Fr(F_\alpha)=\bar{F}_\alpha \cap \overline{X-A_\alpha}$.

4) For the subset H_α of a topological space X, $\bar{H}_\alpha$ denotes the closure of H_α in X.

5) For the collection $\mathfrak{H}$ of subsets of a topological space X, $\bar{\mathfrak{H}}$ denotes the collection $\{\bar{U} \mid U \in \mathfrak{H}\}$.

get Theorem 2.

Theorem 3. *Let* $\{G_\alpha \mid \alpha \in A\}$ *be a locally finite open covering of a regular* T_1*-space* X *and each* G_α *be a strongly paracompact subspace such that* $Fr(G_\alpha)$ *has the Lindelöf property. If* $G_\alpha \cap G_\beta$ *has the locally Lindelöf property for each* $\alpha, \beta \in A$ *with* $\alpha \neq \beta$, *then the whole space* X *is strongly paracompact.*

Proof. By A. Okuyama ([3]; Theorem 1), X is normal and therefore X is strongly paracompact by the above Theorem 2.

Definition. Let x be the point of a topological space X. X is said to have *the locally peripherally Lindelöf property at* x if any neighborhood of x contains an open neighborhood of x whose boundary has the Lindelöf property,[6] that is, there exists an open neighborhood base of x consisting of the open sets whose boundaries have the Lindelöf property (see [4]).

By the use of the locally peripherally Lindelöf property, we will show some theorems for the unions of the strongly paracompact open subspaces as under.

For this purpose, we will prove the following lemma in advance.

Lemma. *Let* F *be the strongly paracompact closed subspace of a topological space* X *with the Lindelöf boundary in* X. *Then an arbitrary open covering of* F *is refined by the star finite open covering of* F *such that* $Fr(F)$ *intersects at most countably many elements of it (where "open" means the open set in* F*).*

Proof. Let $\mathfrak{A}$ be an any open covering of F, and therefore $\mathfrak{A}$ is refined by the star finite (in F) covering $\mathfrak{W}=\{W_\alpha \mid \alpha \in A\}$ of F by the strong paracompactness of F. From the normality of F, we have an open (in F) covering $U=\{U_\alpha \mid \alpha \in A\}$ of F such that:

$U_\alpha \subset \bar{U}_\alpha$ (closure in F and hence in X) $\subset W_\alpha$ for any $\alpha \in A$.

Then $\mathfrak{U}$ is a star a finite open covering of F and, from the Lindelöf property of $Fr(F)$, it is covered by countably many elements $U_{\alpha_i} \in \mathfrak{U} (i=1, 2, \cdots)$. Then it is easily seen that $\mathfrak{V}=\{V_\alpha \mid \alpha \in A\}$ is the desired covering, where

$$V_\alpha = W_\alpha - \bigcup_{i=1}^{\infty} \bar{U}_{\alpha i} \text{ for } \alpha \in A \text{ with } \alpha \notin \{\alpha_1, \alpha_2, \cdots\},$$

$$V_\alpha = W_\alpha \qquad \text{for } \alpha \in A \text{ with } \alpha = \alpha_j (i=1, 2, \cdots).$$

Theorem 4. *Let* X *be a regular* T_1*-space and* $\{G_i \mid i=1, 2, \cdots, n\}$ *be a finite open covering of* X *consisting of strongly paracompact subspaces. If* $Fr(G_i)$ *is compact and* X *has the locally peripherally Lindelöf at* $Fr(G_i)$ *for each* $i=1, 2, \cdots, n$, *then* X *is strongly paracompact.*

Proof. It is sufficient to prove only in the case of $n=2$. Let $F=\bar{G}_1$. At first we shall prove the strong paracompactness of F.

6) In this case, we say that the set has the Lindelöf boundary.

For this purpose, let $\mathfrak{A}$ be an arbitrary open covering of F as a subspace of X. Since $Fr(\bar{G}_1)\subset Fr(G_1)$ is compact by the hypothesis, we have the finite open collection $\{V_{\alpha_i}\mid i=1,2,\cdots,m\}$ of X each of which has the Lindelöf boundary in X, and such that $\{\bar{V}_{\alpha_i}\cap F\mid i=1,\cdots,m\}$ is a refinement of $\mathfrak{A}$ which covers of $Fr(F)$.

Let $H=F-\bigcup_{i=1}^{m}V_{\alpha i}$, then H is closed in X and $H\subset G_1$, and hence H is strongly paracompact.

Now

$$\begin{aligned}Fr(H)&=\bar{H}\cap\overline{X-H}=\bar{H}\cap(\overline{X-F}\cup\overline{F-H})\\&=(\bar{H}\cap\overline{X-F})\cup(\bar{H}\cap\overline{F-H}).\end{aligned}$$

In this place,

$$\bar{H}\cap\overline{X-F}\subset\bar{H}=H,$$

$$\bar{H}\cap\overline{X-F}\subset\bar{F}\cap\overline{X-F}=Fr(F)\subset\bigcup_{i=1}^{m}V_{\alpha i},$$

and therefore

$$\bar{H}\cap\overline{X-F}\subset H\cap\Big(\bigcup_{i=1}^{m}V_{\alpha_j}\Big)=\phi,$$

and then,

$$\begin{aligned}Fr(H)&=\bar{H}\cap\overline{F-H}\subseteq\overline{F-\bigcup_{i=1}^{m}V_{\alpha_i}}\cap\overline{\bigcup_{i=1}^{m}V_{\alpha_i}}\subset\overline{X-\bigcup_{1}^{m}V_{\alpha_i}}\cap\overline{\bigcup_{i=1}^{m}V_{\alpha_i}}\\&=Fr\Big(\bigcup_{i=1}^{m}V_{\alpha_i}\Big)\subseteq\bigcup_{i=1}^{m}Fr(V_{\alpha_i}).\end{aligned}$$

Consequently, from the Lindelöf property of $Fr(V_{\alpha_i})$ for each $i=1,2,\cdots,m$, we have the Lindelöf property of $Fr(H)$, and hence, by the above lemma, it is easily seen the H has the locally finite, star finite closed covering $\mathfrak{U}$ as a refinement of $\mathfrak{A}\cap H$,[7] such that $Fr(H)$ intersects at most countably many elements of it.

Now, let $\mathfrak{B}=\{\bar{V}_{\alpha_i}\cap F\mid i=1,2,\cdots,m\}\cup\mathfrak{U}$, then it is easily seen that $\mathfrak{B}$ is a locally finite closed covering of F and a refinement of $\mathfrak{A}$. In order to show the star countability of $\mathfrak{B}$, we need prove only $\overline{V_{\alpha_{i_0}}}\cap H\subset Fr(H)$. Not suppose, and then some point $x\in\overline{V_{\alpha_{i_0}}}\cap H-Fr(H)$. Since $x\in H-Fr(H)$, some open set $U(x)$ containing x does not intersect with $\overline{X-H}$, that is,

$$\phi=U(x)\cap(X-H)=U(x)\cap\Big(\overline{X-F}\cup\bigcup_{i=1}^{m}V_{\alpha_i}\Big).$$

On the other hand, $x\in H\subset F-Fr(F)$, we can suppose $U(x)\cap\overline{X-F}=\phi$, and then $\phi=U(x)\cap\overline{\bigcup_{i=1}^{m}V_{\alpha_i}}\supseteq U(x)\cap\overline{V_{\alpha_{i_0}}}$, this is contrary to $x\in\overline{V_{\alpha_{i_0}}}$. From the above fact, any open covering of F has the locally finite star countable closed covering of F as a refinement and therefore F is strongly paracompact by Yu. M. Smirnov ([5]; Theorem 1).

7) For the subset H of the topological space X and the collection $\mathfrak{A}$ of subsets of X, $\mathfrak{A}\cap H$ denotes the collection $\{A\cap H\mid A\cap\mathfrak{A}\}$.

Secondly let $F_2=\overline{G_2-G_1}$, then $F_2\subset G_2$ and hence F_2 is strongly paracompact closed subspace of X. Since $Fr(F_2)=Fr(G_1)$ is Lindelöf, $\{F_1, F_2\}$ is a closed covering of X each of which is a strongly paracompact closed subspace with the locally Lindelöf boundary and consequently, we get Theorem 4, by Theorem 1 or V. Trnkova ([6]; Proposition 5).

In the same way as the above proof, we can prove the following:

Theorem 5. *Let $\{A, B\}$ be the covering of X such that A is compact and B is the strongly paracompact open subspace of X. If X has the locally peripherally Lindelöf at A, then X is strongly paracompact.*

At last, we have the following theorem which is a generalization of Theorem 4:

Theorem 6. *Let $\{G_\alpha \mid \alpha\in A\}$ be a locally finite open covering of a regular T_1-space X each of which is a strongly paracompact open subspace with the compact boundary in X. If X has the locally peripherally Lindelöf at $\bigcup_{\alpha\in A} Fr(G_\alpha)$, then X is strongly paracompact.*

Proof. It is easily seen from the compactness of $Fr(G_\alpha)$ for each α.

Remark. (1) Theorem 3 (resp. Theorem 2) is a generalization of ([1]; Theorem 5) (resp. [7]; Theorem 4).

(2) In Theorem 5, we cannot drop the hypothesis of the locally peripherally Lindelöf property. In fact, let X be the Euclidean plane set and ρ be a usual metric on X. We shall define the following other metric d on X;

$d(x, y)=\rho(x, 0)+\rho(0, y)$ for each x, $y\in X$, where 0 is the original point of X. Then it is already known that this metric space (X, d) is not strongly paracompact. Now let $A=\{0\}$ and $B=X-\{0\}$, then A is compact and B has the discrete open covering, as a subspace of X, each of which is the metric space with the countable open base (see [5]).

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