# 77. On Submanifolds in Spaces of Constant and Constant Holomorphic Curvatures

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**1. Fundamental formulas.** Let $M$ and $\bar{M}$ be two Riemannian manifolds of dimension $n$ and $n+m$ respectively, with $M$ immersed in $\bar{M}$. We shall denote $\langle\ ,\ \rangle$ the Riemannian metric of $\bar{M}$ and $\bar{\nabla}$ the Riemannian connection of $\bar{M}$ associated with this metric. Let us also denote $\langle\ ,\ \rangle$ the induced Riemannian metric of $M$. Let $V(M)$ be the ring of the differentiable vector fields on $M$, $NV(M)$ be the collection of normal vector fields to $M$ defined on a proper open subset of $M$, which is spanned by mutually orthogonal $m$ unit normal vector fields $C_1, \cdots, C_m$.

Let $$p:\ V(M)+NV(M)\to V(M)$$
be a natural projection.

For $X$ in $V(M)$, we put

(1.1) $$p\bar{\nabla}_X C_i = -A_i X. \qquad (i=1, \cdots, m)$$

**Proposition 1.1.** For $X, Y$ in $V(M)$, we have

(1.2) $$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{m} \langle A_i X, Y\rangle C_i \quad \text{where} \quad \nabla_X Y \quad \text{in} \quad V(M).$$

(1.3) $\nabla$ is a Riemannian connection of $M$ associated with the induced Riemannian metric and $A_i$ are self-adjoint (1, 1) type tensors.

**Proof.** We may set

(1.4) $$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{m} f_i C_i.$$

Then, since $\langle Y, C_i\rangle = 0$, differentiating covariantly, we get

(1.5) $$\langle \bar{\nabla}_X Y, C_i\rangle + \langle Y, \bar{\nabla}_X C_i\rangle = 0.$$

Substituting (1.4) into (1.5) leads to

(1.6) $$f_i = \langle A_i X, Y\rangle.$$

The properties of (1.3) can be easily checked. Q.E.D.

Let $\{E_1, \cdots, E_n\}$ be an orthonormal basis on an open subset of $M$. We put

(1.7) $$H = \sum_{i=1}^{m} (\operatorname{tr} A_i) C_i$$

where tr denotes the trace, $\operatorname{tr} A_i = \sum_{\alpha=1}^{n} \langle A_i E_\alpha, E_\alpha\rangle$. $H$ is called the mean curvature vector field of $M$. A submanifold $M$ is called minimal if $\operatorname{tr} A_i = 0$, totally geodesic if $A_i = 0$ and totally umbilical if $\langle A_i X, X\rangle$

$=\langle A_iY, Y\rangle$ for all $X$ and $Y$ in $V(M)$ with $\|X\|=\|Y\|$.

**Proposition 1.2.** For $X$, $Y$, $W$, and $U$ in $V(M)$ we have

$$pR(W, X)Y=r(W, X)Y+\sum_{i=1}^{m}\{\langle A_iW, Y\rangle A_iX-\langle A_iX, Y\rangle A_iW\} \tag{1.8}$$

$$\begin{aligned} R(U, Y, W, X)&\equiv\langle \bar{R}(W, X)Y, U\rangle \\ &=r(U, Y, W, X)+\sum_{i=1}^{m}\{\langle A_iW, Y\rangle\langle A_iX, U\rangle-\langle A_iX, Y\rangle\langle A_iW, U\rangle\} \end{aligned} \tag{1.9}$$

$$\begin{aligned} R(Y, W)&\equiv\sum_{\alpha=1}^{n}R(E_\alpha, Y, W, E_\alpha) \\ &=r(Y, W)-\langle\bar{\nabla}_WH, Y\rangle-\sum_{i=1}^{m}\sum_{\alpha=1}^{n}\langle A_iY, E_\alpha\rangle\langle A_iW, E_\alpha\rangle \end{aligned} \tag{1.10}$$

$$\begin{aligned} R(Y)&\equiv R(Y, Y) \\ &=r(Y)+\sum_{i=1}^{m}(\mathrm{tr}\, A_i)\langle A_iY, Y\rangle-\sum_{i=1}^{m}\sum_{\alpha=1}^{n}\langle A_iY, E_\alpha\rangle^2 \end{aligned} \tag{1.11}$$

$$\begin{aligned} R&\equiv\sum_{\beta=1}^{m}R(E_\beta) \\ &=r+\sum_{i=1}^{m}(\mathrm{tr}\, A_i)^2-\sum_{i=1}^{m}\sum_{\alpha,\beta=1}\langle A_iE_\alpha, E_\beta\rangle^2 \end{aligned} \tag{1.12}$$

where $\bar{R}(W, X)Y$ and $r(W, X)Y$ are the curvature tensor fields of $\bar{M}$ and $M$ respectively, $r(Y, W)$ is the Ricci curvature and $r$ is the scalar curvature of $M$. We also put $\mathrm{r}(Y)=r(Y, Y)$.

**Proof.** Differentiating covariantly (1.2) we have

$$\begin{aligned} \bar{\nabla}_W\bar{\nabla}_XY=\nabla_W\nabla_XY+\sum_{i=1}^{m}\{&\langle A_iW, \nabla_XY\rangle C_i+\bar{\nabla}_W(\langle A_iX, Y\rangle)C_i \\ &+\langle A_iX, Y\rangle\bar{\nabla}_WC_i\}. \end{aligned} \tag{1.13}$$

Hence

$$p\bar{\nabla}_W\bar{\nabla}_XY=\nabla_W\nabla_XY-\sum_{i=1}^{m}\langle A_iX, Y\rangle A_iW. \tag{1.14}$$

Thus

$$\begin{aligned} p\bar{R}(W, X)Y&=p(\bar{\nabla}_W\bar{\nabla}_XY-\bar{\nabla}_X\bar{\nabla}_WY-\bar{\nabla}_{[W,X]}Y \\ &=r(W, X)Y+\sum_{i=1}^{m}\{\langle A_iW, Y)A_iX-\langle A_iX, Y\rangle A_iW\} \end{aligned}$$

which is the equation of Gauss.

For the proof of (1.10), we have

$$\begin{aligned} R(Y, W)&=\sum_{\alpha=1}^{n}r(E_\alpha, Y, W, E_\alpha)+\sum_{i=1}^{m}\sum_{\alpha=1}^{n}\{\langle A_iW, Y\rangle\langle A_iE_\alpha, E_\alpha\rangle \\ &\quad-\langle A_iE_\alpha, Y\rangle\langle A_iE_\alpha, W\rangle\} \\ &=r(Y, W)+\sum_{i=1}^{m}\langle A_iW, Y\rangle(\mathrm{tr}\, A_i) \\ &\quad-\sum_{i=1}^{m}\sum_{\alpha=1}^{n}\langle A_iE_\alpha, Y\rangle\langle A_iE_\alpha, W\rangle. \end{aligned} \tag{1.15}$$

On the other hand, differentiating (1.7) and making an inner product $\bar{\nabla}_WH$ with $Y$, we get

(1.16) $$\langle \bar{\nabla}_W H, Y\rangle = -\sum_{i=1}^{m} \langle A_i W, Y\rangle(\mathrm{tr}\, A_i).$$

Substituting this into (1.15), we have the required (1.10). Q.E.D.

**Theorem 1.3.** Let $M$ be a minimal submanifold. Then

(1.17) $$R(Y, W) = r(Y, W) - \sum_{i=1}^{m}\sum_{\alpha=1}^{n} \langle A_i Y, E_\alpha\rangle\langle A_i W, E_\alpha\rangle$$

(1.18) $$R(Y) \leqq r(Y)$$

and the equality occurs if and only if $M$ is totally geodesic.

**2. Submanifolds in a space of constant curvature.** Let $\bar{M}$ be a space of constant curvature. Then the curvature tensor field of $\bar{M}$ is given by

(2.1) $$\bar{R}(\bar{W}, \bar{X})\bar{Y} = k\{\langle \bar{X}, \bar{Y}\rangle \bar{W} - \langle \bar{W}, \bar{Y}\rangle \bar{X}\}$$

where $\bar{X}, \bar{Y}, \bar{W}$ are in $V(\bar{M})$ and $k$ is a constant.

**Lemma 2.1.** For $Y$ and $W$ in $V(M)$, we have

(2.2) $$\sum_{\alpha=1}^{n} \langle E_\alpha, Y\rangle\langle E_\alpha, W\rangle = \langle Y, W\rangle.$$

**Proposition 2.2.** Let $M$ be a submanifold in a space of constant curvature. Then for $X$, $Y$, $W$, and $U$ in $V(M)$

(2.3) $$r(W, X)Y = k\{\langle X, Y\rangle W - \langle W, Y\rangle X\} - \sum_{i=1}^{m} \{\langle A_i W, Y\rangle A_i X - \langle A_i X, Y\rangle A_i W\}$$

(2.4) $$r(U, Y, W, X) = k\{\langle X, Y\rangle\langle W, U\rangle - \langle W, Y\rangle\langle X, U\rangle\} - \sum_{i=1}^{m} \{\langle A_i W, Y\rangle\langle A_i X, U\rangle - \langle A_i X, Y\rangle\langle A_i W, U\rangle\}$$

(2.5) $$r(Y, W) = (k - kn)\langle W, Y\rangle - \sum_{i=1}^{m} \langle A_i W, Y\rangle(\mathrm{tr}\, A_i) + \sum_{i=1}^{m}\sum_{\alpha=1}^{n} \langle A_i E_\alpha, Y\rangle\langle A_i E_\alpha, W\rangle$$

(2.6) $$r(Y) = (k - kn)\,\| Y \|^2 - \sum_{i=1}^{m} (\mathrm{tr}\, A_i)\langle A_i Y, Y\rangle + \sum_{i=1}^{m}\sum_{\alpha=1}^{n} \langle A_i E_\alpha, Y\rangle^2$$

(2.7) $$r = kn - kn^2 - \sum_{i=1}^{m} (\mathrm{tr}\, A_i)^2 + \sum_{i=1}^{m}\sum_{\alpha,\beta=1}^{n} \langle A_i E_\alpha, E_\beta\rangle^2.$$

**Proof.** For the proof of (2.5), we have

$$\begin{aligned} r(Y, W) &= k\sum_{\alpha=1}^{n}\{\langle E_\alpha, Y\rangle\langle E_\alpha, W\rangle - \langle W, Y\rangle\langle E_\alpha, E_\alpha\rangle\} \\ &\quad - \sum_{i=1}^{m}\sum_{\alpha=1}^{n} \{\langle A_i W, Y\rangle\langle A_i E_\alpha, E_\alpha\rangle - \langle A_i E_\alpha, Y\rangle\langle A_i W, E_\alpha\rangle\} \\ &= (k - kn)\langle W, Y\rangle - \sum_{i=1}^{m} \langle A_i W, Y\rangle(\mathrm{tr}\, A_i) + \sum_{i=1}^{m}\sum_{\alpha=1}^{n} \langle A_i E_\alpha, Y\rangle\langle A_i E_\alpha, W\rangle \end{aligned}$$

by the above lemma. Q.E.D.

**Proposition 2.3.** Let $M$ be a submanifold in a space of constant curvature. Then

(2.8) $$K = k - \sum_{i=1}^{m} \langle A_i X, Y\rangle^2 + \sum_{i=1}^{m} \langle A_i X, X\rangle\langle A_i Y, Y\rangle$$

where $K$ is the sectional curvature of $M$ spanned by an orthonormal basis $\{X, Y\}$.

**Proof.** It is clear from the definition of the sectional curvature

$$K=r(X, Y, X, Y). \qquad \text{Q.E.D.}$$

**Theorem 2.4.** Let $M$ be a minimal submanifold in a space of constant curvature. Then

$$r(Y)\geqq k(1-n)\|Y\|^2 \tag{2.9}$$

$$r\geqq k(1-n)n \tag{2.10}$$

and the both equalities occur if and only if $M$ is totally geodesic.

**Theorem 2.5.** Let $M$ be a totally umbilical submanifold in a space of constant curvature. Then

$$r\geqq(1-n)nK \tag{2.11}$$

and the equality occurs if and only if $M$ is totally geodesic.

**Proof.** Since $M$ is totally umbilical, (2.9) reduces to

$$K=k-\sum_{i=1}^{m}\langle A_iX, Y\rangle^2+\sum_{i=1}^{m}\langle A_iX, X\rangle^2 \quad \text{for} \quad \langle X, Y\rangle=0. \tag{2.12}$$

We put

$$a=\sum_{i=1}^{m}\langle A_iX, X\rangle^2. \tag{2.13}$$

Then, since $\sum_{i=1}^{m}(\mathrm{tr}\, A_i)^2=na$, (2.7) becomes

$$r=kn-kn^2-na+\sum_{i=1}^{m}\sum_{\alpha,\beta=1}^{n}\langle A_iE_\alpha, E_\beta\rangle^2. \tag{2.14}$$

Since $\langle E_\alpha, E_\beta\rangle=0$ for $\alpha\neq\beta$, we get

$$\sum_{i=1}^{m}\langle A_iE_\alpha, E_\beta\rangle^2=k-K+a\,. \tag{2.15}$$

Hence, $\sum_{i=1}^{m}\sum_{\alpha,\beta=1}^{n}\langle A_iE_\alpha, E_\beta\rangle^2=n(n-1)(k-K+a)+na.$

Thus, substituting this into (2.14) we have

$$n(n-1)a=r+(n-1)nK. \tag{2.16}$$

Since $a\geqq 0$, we get $\quad r\geqq(1-n)nK.$ Q.E.D.

**Remark.** If $M$ is a totally umbilical submanifold in a Riemannian manifold $\bar{M}$ with the sectional curvature $K$, then

$$R-r\leqq n(n-1)(K-K).$$

**3. Submanifolds in a space of constant holomorphic curvature.** Let $\bar{M}$ be a space of constant holomorphic curvature. Then the curvature tensor fields of $\bar{M}$ is given by

$$\begin{aligned}\bar{R}(\bar{W}, \bar{X})\bar{Y}=k\{&\langle\bar{X}, \bar{Y}\rangle\bar{W}-\langle\bar{W}, \bar{Y}\rangle\bar{X}+\langle J\bar{X}, \bar{Y}\rangle J\bar{W}\\ &-\langle J\bar{W}, \bar{Y}\rangle J\bar{X}-2\langle J\bar{W}, \bar{X}\rangle J\bar{Y}\}\end{aligned} \tag{3.1}$$

where $\bar{X}$, $\bar{Y}$, and $\bar{W}$ are in $V(\bar{M})$, $k$ is a constant and $J$ is the Kähler structure of $\bar{M}$. We may assume the Riemannian metric to be Hermitian.

For $X$ in $V(M)$, we put

$$JX = TX + NX \tag{3.2.}$$

where $TX$ is in $V(M)$ and $NX$ is in $NV(M)$.

A submanifold $M$ in an almost complex manifold is called invariant if $JX = TX$, anti-holomorphic if $JX = NX$.

**Proposition 3.1.** Let $M$ be a submanifold in a space of constant holomorphic curvature. Then, for $X$, $Y$, $W$, and $U$ in $V(M)$

$$\begin{aligned} r(W, X)Y = k\{&\langle X, Y\rangle W - \langle W, Y\rangle X + \langle TX, Y\rangle TW \\ &- \langle TW, Y\rangle TX - 2\langle TW, X\rangle TY\} \\ &- \sum_{i=1}^{m}\{\langle A_i W, Y\rangle A_i X - \langle A_i X, Y\rangle A_i W\} \end{aligned} \tag{3.3}$$

$$\begin{aligned} r(U, Y, W, X) = k\{&\langle X, Y\rangle\langle W, U\rangle - \langle W, Y\rangle\langle X, U\rangle \\ &+ \langle TX, Y\rangle\langle TW, U\rangle - 2\langle TW, X\rangle\langle TX, U\rangle\} \\ &- \sum_{i=1}^{m}\{\langle A_i W, Y\rangle\langle A_i X, U\rangle - \langle A_i X, Y\rangle\langle A_i W, U\rangle\} \end{aligned} \tag{3.4}$$

$$\begin{aligned} r(Y, W) = k(1-n)\langle W, Y\rangle - 3k\langle TW, TY\rangle - \sum_{i=1}^{m}\langle A_i W, Y\rangle (\operatorname{tr} A_i) \\ + \sum_{i=1}^{m}\sum_{\alpha=1}^{n}\langle A_i E_\alpha, Y\rangle\langle A_i E_\alpha, W\rangle \end{aligned} \tag{3.5}$$

$$r(Y) = (k - kn)\|Y\|^2 - 3k\|TY\|^2 - \sum_{i=1}^{n}(\operatorname{tr} A_i)^2 + \sum_{i=1}^{m}\sum_{\alpha,\beta=1}^{n}\langle A_i E_\alpha, E_\beta\rangle \tag{3.6}$$

and for an orthonormal basis $\{X, Y\}$,

$$K = k(1 + 3\langle TY, X\rangle^2) - \sum_{i=1}^{m}\langle A_i X, Y\rangle^2 + \sum_{i=1}^{m}\langle A_i X, X\rangle\langle A_i Y, Y\rangle. \tag{3.7}$$

**Proof.** For the proof of (3.5), we use the fact

$$\sum_{\alpha=1}^{n}\langle TY, E_\alpha\rangle\langle TW, E_\alpha\rangle = \langle TY, TW\rangle$$

and the identity $\langle JX, Y\rangle = -\langle X, JY\rangle$. Q.E.D.

**Theorem 3.2.** Let $M$ be a totally geodesic submanifold in a space of constant holomorphic curvature. Then for $Y$ with $\|Y\|^2 = b$, we have

$$(k - kn)b \geqq r(Y) \geqq (-2k - kn)b \quad \text{for } k > 0 \tag{3.9}$$

$$(k - kn)b \leqq r(Y) \leqq (-2k - kn)b \quad \text{for } k < 0 \tag{3.10}$$

and the equality $r(Y) = bk - bkn$ occurs if and only if $M$ is anti-holomorphic and $r(Y) = -2bk - bkn$ occurs if and only if $M$ is invariant.

**Proof.** Since $M$ is totally geodesic, (3.6) reduces to

$$r(Y) = bk - bkn - 3k\|TY\|^2. \tag{3.11}$$

Since $0 \leqq \|TY\|^2 \leqq b$, we have (3.9) for $k > 0$ and (3.10) for $k < 0$. If $M$ is invariant, then $\|Y\|^2 = \langle JY, JY\rangle = \|TY\|^2$, which implies $r(Y) = -2bk - bkn$, and vice versa. Q.E.D.

**Theorem 3.3.** Notations being as above. Then we have

$$kn - kn^2 \geqq r \geqq -2kn - kn^2 \quad \text{for } k > 0 \tag{3.12}$$

$$kn - kn^2 \leqq r \leqq -2kn - kn^2 \quad \text{for } k < 0. \tag{3.13}$$

Especially, if $\langle TX, TY\rangle = 0$ for all orthogonal pairs $\{X, Y\}$ then the equality $r = kn - kn^2$ occurs if and only if $M$ is anti-holomorphic, $r = -2kn - kn^2$ occurs if and only if $M$ is invariant.

**Proof.** If $M$ is invariant, then $\|TE_\alpha\|^2=1$, which gives $r=-2kn-kn^2$. Convercely, if $r=-2kn-kn^2$, then $\|TE_\alpha\|^2=1$. Hence, for $X=\sum_{\alpha=1}^{n} f_\alpha E_\alpha$

$$\|TX\|^2=\sum_{\alpha=1}^{n} f_\alpha^2\langle TE_\alpha, TE_\alpha\rangle+\sum_{\alpha,\beta=1\ (\alpha\neq\beta)}^{n} f_\alpha f_\beta\langle TE_\alpha, TE_\beta\rangle \tag{3.15}$$
$$=\sum_{\alpha=1}^{n} f_\alpha^2=\|X\|^2=\|JX\|^2$$

that is $M$ is invariant. Q.E.D.

As was proved in Theorem 2.6, we can state the following Theorem 3.4.

**Theorem 3.4.** Let $M$ be a totally umbilical submanifold in a space of constant holomorphic curvature. Then

$$r\geqq n(1-n)K \tag{3.16}$$

and the equality occurs if and only if $M$ is totally geodesic.

**Proof.** Since $M$ is totally umbilical, (5.8) reduces to

$$K=k(1+3\langle TY, X\rangle^2)-\sum_{i=1}^{m}\langle A_iX, Y\rangle^2+\sum_{i=1}^{m}\langle A_iX, X\rangle^2 . \tag{3.17}$$

Thus applying the similar calculations used in Theorem 2.5 we have the required (3.16).

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