66. On Tabooistic Treatment of Proposition Logics

By Katuzi ONO
Mathematical Institute, Nagoya University, Nagoya

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1. The purpose of this short note is to remark that the tabooistic treatment of formal theories introduced in my paper [1] can be nicely applied to dealing with axiomatizable proposition logics which are stronger than or equivalent to the *generalized minimal proposition logic*. The *minimal proposition logic* ***LMS*** has $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction), and $\sim$ (negation) as its logical constants and is characterized by the following inference rules:

F: $\mathfrak{A}$ *is deducible from* $\mathfrak{A}$.

I: $\mathfrak{A}$ *is deducible from* $\mathfrak{B}$ *and* $\mathfrak{B}\rightarrow\mathfrak{A}$.

I*: $\mathfrak{A}\rightarrow\mathfrak{B}$ *is deducible from the fact that* $\mathfrak{B}$ *is deducible from* $\mathfrak{A}$.

C: $\mathfrak{A}$ *as well as* $\mathfrak{B}$ *is deducible from* $\mathfrak{A}\wedge\mathfrak{B}$.

C*: $\mathfrak{A}\wedge\mathfrak{B}$ *is deducible from* $\mathfrak{A}$ *and* $\mathfrak{B}$.

D: $\mathfrak{A}$ *is deducible from* $\mathfrak{B}\vee\mathfrak{C}$, $\mathfrak{B}\rightarrow\mathfrak{A}$, *and* $\mathfrak{C}\rightarrow\mathfrak{A}$.

D*: $\mathfrak{A}\vee\mathfrak{B}$ *is deducible from* $\mathfrak{A}$ *as well as from* $\mathfrak{B}$.

N: $\sim\mathfrak{A}$ *stands for* $\mathfrak{A}\rightarrow\curlywedge$, *where* $\curlywedge$ *is a proposition constant.*

In *generalized formalism* of proposition logic where we adopt the universal quantification ranging over proposition variables $x, y, \cdots$, we can reformulate the minimal proposition logic as the logic ***LMS**** characterized by the following inference rules and axioms:

Inference rules: ***F***, ***I***, ***I****, and

$\bar{U}$: $\mathfrak{A}(\mathfrak{F})$ *is deducible from* $(x)\mathfrak{A}(x)$, *where* $\mathfrak{F}$ *is a propositional expression containing no quantification.*

Axioms:

c1: $(x)(y)(x\wedge y\rightarrow x)$, **c2**: $(x)(y)(x\wedge y\rightarrow y)$,

c*: $(x)(y)(x\rightarrow(y\rightarrow x\wedge y))$,

d: $(x)(y)(z)(y\vee z\rightarrow((y\rightarrow x)\rightarrow((z\rightarrow x)\rightarrow x)))$,

d*1: $(x)(y)(x\rightarrow x\vee y)$, **d*2**: $(x)(y)(y\rightarrow x\vee y)$,

n1: $(x)(\sim x\rightarrow(x\rightarrow\curlywedge))$, **n2**: $(x)((x\rightarrow\curlywedge)\rightarrow\sim x)$.

Any proposition $\mathfrak{A}$ containing no quantification is provable in ***LMS*** if and only if $\mathfrak{A}$ is provable in the *generalized minimal proposition logic* ***LMS****.

In generalizing the notion "intermediate proposition logic", I will call any proposition logic ***L***, in generalized formalism or not, an intermediate proposition logic if and only if every provable proposition in

LMS* containing no quantification is provable in ***L***.

It is hard to introduce intermediate proposition logics by finite numbers of axioms over the minimal proposition logic ***LMS*** in non-generalized formalism, but a quite extensive class of intermediate proposition logics can be introduced, each by a finite number of axioms over the minimal proposition logic ***LMS**** in the generalized formalism. For example, we can express *tertium non datur* in the single axiom

$$(x)(x \vee \sim x)$$

in ***LMS****, but we can express it only by the axiom schema

$$X \vee \sim X$$

in ***LMS***.

2. Propositions would be indicated by indices which can be regarded as objects. I will indicate the propositions $P, Q, \cdots$ by the indices $p, q, \cdots$, which are objects. Namely, $P, Q, \cdots$ can be denoted in the forms $\Phi(p), \Phi(q), \cdots$ in taking up a predicate Φ.

In the generalized proposition logics, we can regard the bound variables in quantifiers as object variables ranging over the index domain, and any other variable x as the abbreviation of the propositional expression $\Phi(x)$. Then, finitely axiomatizable intermediate proposition logics over ***LMS**** turn out to be very close to formal theories standing on the minimal logic ***LM***. The only trouble is the inference rule ***U***, which states that $(x)\mathfrak{A}(x)$ implies $\mathfrak{A}(\mathfrak{F})$ for propositional expressions $\mathfrak{F}$.

In reality, however, we can replace the inference rule $\bar{U}$ by the inference rule

U: $(x)\mathfrak{A}(x)$ *implies* $\mathfrak{A}(f)$ *for any proposition variable* f

and the following axioms:

$(x)(y)(\exists z)(z \equiv (x \rightarrow y))$,

$(x)(y)(\exists z)(z \equiv x \wedge y)$,

$(x)(y)(\exists z)(z \equiv x \vee y)$,

$(x)(\exists z)(z \equiv \sim x)$

in the minimal logic ***LM***. Here, $\mathfrak{F} \equiv \mathfrak{G}$ stands for $(\mathfrak{F} \rightarrow \mathfrak{G}) \wedge (\mathfrak{G} \rightarrow \mathfrak{F})$, as usual.

Accordingly, an extensive class of intermediate proposition logics, *i.e.* the class of finitely axiomatizable proposition logics over ***LMS****, can be transformed into axiomatic formal theories standing on the minimal logic ***LM***.

3. According to my paper [1], any axiomatic formal theory standing on the minimal logic ***LM*** can be transformed into a tabooistic formal theory. So,

Theorem. *Any intermediate proposition logic which is finitely* *axiomatizable on the minimal proposition logic **LMS**∗ in the generalized formalism can be reformulated into a tabooistic formal theory.*

In other words,

Theorem. *If any intermediate proposition logic **L** can be introduced by assuming a finite number of axioms over the minimal proposition logic **LMS**∗ in the generalized formalism, we can define the logical constants "$\wedge$", "$\vee$", and "$\sim$" in terms of "$\rightarrow$" and "( )" in the primitive logic **LO** in such way that any proposition-logical proposition is provable in the proposition logic **L** if and only if it is provable in the primitive logic **LO**.*

Reference

[1] K. Ono: On formal theories (to appear in Nagoya Math. J.).