150. *Weak Convergence of the Isotropic Scattering Transport Process with One Speed in the Plane to Brownian Motion*

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Let us consider a particle moving in the d-dimensional Euclidian space R^d. It travels in a straight line with constant speed and after some random time it undergoes scattering which changes its moving-direction and so, after scattering, continues to move as if it starts afresh. Now let $P(t, x, \Gamma)$ be the probability that we can find the particle in a region Γ at time t when it starts from x. In 2-dimensional isotropic case, Monin [4] has obtained the explicite formula of $P(t, x, \cdot)$ and has shown that it converges to 2-dimensional Gaussian distribution as $t\to\infty$. On the other hand, in one-dimensional isotropic case, Ikeda and Nomoto [3] have proved that $P(t, x, \cdot)$ converges to a Gaussian distribution as the speed of the particle tends to infinity in an appropriate manner and moreover they have shown that the measure on the space of trajectories of the motion also converges to the Wiener measure.

The purpose of this paper is to prove that the same result to them is also valid for the two-dimensional case.

1. Notations and definitions. Let $\Theta=[\theta(t), +\infty, \mathcal{N}_t, P.]$ be a right continuous jump process on the state space $[-\pi, \pi)$, which is identified to the unit circle S^1. Also let τ be the first jumping time of Θ, i.e., $\tau=\inf\{t : \theta(t)\neq\theta(0)\}$. We assume that the following conditions be satisfied:

(i) $P\{\tau>t\}=e^{-c^2t}$, $c>0$ constant,

(ii) $P\{\theta(\tau)\in\Gamma\}=|\Gamma|/2\pi$,

where $|\Gamma|$ denotes the Lebesgue measure of the set Γ.

The formula

$$A(\mathrm{t})=\left(\int_0^t\cos\theta(s)ds, \int_0^t\sin\theta(s)ds\right) \tag{1.1}$$

defines on R^2-valued continuous additive functional of Θ. Let $E=R^2\times S^1$ be the product space of R^2 and S^1, and $\mathcal{B}(E)$ be the topological Borel field of E. For each point $(x, \theta)\in E$, we define the following:

$$X^{(x,\theta)}(t)=(x+cA(t), \theta(t)), \tag{1.2}$$

$$P^{(x,\theta)}\{X^{(x,\theta)}(t)\in B\}=P\{X^{(x,\theta)}(t)\in B\}, \qquad B\in\mathcal{B}(E), \tag{1.3}$$

$$\mathcal{M}_t^{(x,\theta)}=\mathcal{N}_t. \tag{1.4}$$

Then it is not difficult to see that $X^{(x,\theta)}=[X^{(x,\theta)}(t), +\infty, \mathcal{M}_t^{(x,\theta)}, P^{(x,\theta)}, (x, \theta)\in E]$ is a system of Markov family of random functions so that there corresponds the Markov process $X=[X(t)=(x(t), \theta(t)), +\infty, \mathcal{M}_t, P_{(x,\theta)}, (x, \theta)\in E]$ (cf. [1]).

Definition 1.1. We call the Markov process X *the isotropic scattering transport process with one speed c*, or simply *the transport process with speed c.*

2. Characteristic function and convergence in distribution. For each $x\in R^2$ we define a measure $P_x(\cdot)$ by

$$P_x\{x(t)\in\Gamma\}=\int_{-\pi}^{\pi} P_{(x,\theta)}\{(x(t), \theta(t))\in \Gamma\times S^1\}d\theta,\ \Gamma\in\mathcal{B}(R^2) \tag{2.1}$$

and consider the stochastic process $[x(t), t\geq 0, P_x, x\in R^2]$, which describes the trajectories of moving particles. Now, let $\varphi^c(\alpha_1, \alpha_2 : t : x)$ be the characteristic function of $[x(t), P_x]$, i.e.

$$\varphi^c(\alpha_1, \alpha_2 : t : x)=E_x[\exp\{i(\alpha_1 x_1(t)+\alpha_2 x_2(t))\}], \tag{2.2}$$

where $x(t)=(x_1(t), x_2(t))$ and $E_x[\cdot]$ is the expectation with respect to $P_x(\cdot)$.

Availing the Kac's formula, we have

Lemma 1.1. (i) $\varphi^c(\alpha_1, \alpha_2 : t : x)=\exp\{i(\alpha_1 x_1+\alpha_2 x_2)\}\varphi^c(\alpha_1, \alpha_2 : t : \mathbf{0})$, *where* $x=(x_1, x_2)$.

(ii) $\varphi^c(\alpha_1, \alpha_2 : t : \mathbf{0})$ *is the unique solutions of the integral equation*:

$$\begin{aligned} U(t)=&e^{-c^2t}J_0(c\sqrt{(\alpha_1^2+\alpha_2^2)}t)\\ &+c^2\int_0^t e^{-c^2u}J_0(c\sqrt{(\alpha_1^2+\alpha_2^2)}u)U(t-u)du, \end{aligned} \tag{2.3}$$

where $J_n(\cdot)$ *is the usual Bessel function of order* n $(n=0, \pm 1, \pm 2, \cdots)$ (*cf.* [5]).

Lemma 1.2. (i) $\varphi^c(\alpha_1, \alpha_2 : t : \mathbf{0})$

$$\begin{aligned} &=\frac{ce^{-c^2t}}{c^2-(\alpha_1^2+\alpha_2^2)}\sin h\{c\sqrt{c^2-(\alpha_1^2+\alpha_2^2)}t\}\\ &\quad+e^{-c^2t}[\sum_{n=0}^{\infty}\frac{2^n n!(c^3t)^n}{(2n)!(\alpha_1^2+\alpha_2^2)^{n/2}}J_n(c\sqrt{(\alpha_1^2+\alpha_2^2)}t)]. \end{aligned}$$

(ii) $$\begin{aligned}\varphi^c(\alpha_1, \alpha_2 : t : \mathbf{0})=&\frac{2ce^{-c^2t}}{c^2-(\alpha_1^2+\alpha_2^2)}\sin h\{c\sqrt{c^2-(\alpha_1^2+\alpha_2^2)}t\}\\ &+0\left(\frac{(\alpha_1^2+\alpha_2^2)e^{-c^2t}}{c^2-(\alpha_1^2+\alpha_2^2)}\right)\qquad (c\to\infty).\end{aligned}$$

Proof. Let $\Phi(s)$ be the Laplace transform of φ^c as the function of t. Then we get from the integral equation (2.3)

$$\Phi(s)=\frac{c^2}{\sqrt{(s+c^2)^2+c^2(\alpha_1^2+\alpha_2^2)}}+\frac{c^2}{\sqrt{(s+c^2)^2+c^2(\alpha_1^2+\alpha_2^2)}}\Phi(s). \tag{2.4}$$

Therefore,

$$\Phi(s)=\frac{c^2}{(s+c^2)^2+c^2(\alpha_1^2+\alpha_2^2)-c^4} \tag{2.5}$$

$$+\sum_{m=0}^{\infty} c^{4m}\{(s+c^2)^2+c^2(\alpha_1^2+\alpha_2^2)\}^{-(2m+1)/2}.$$

$$\Phi(s)=\frac{c^2}{(s+c^2)^2+c^2(\alpha_1^2+\alpha_2^2)-c^4}+\frac{(s+c^2)^2+c^2(\alpha_1^2+\alpha_2^2)}{(s+c^2)^2+c^2(\alpha_1^2+\alpha_2^2)-c^4}. \tag{2.6}$$

After elementary but tedious calculations of the inversion formula of the Laplace transform of $\Phi(s)$, we obtain the expressions of (i) and (ii) of φ^c.

Now let $B=[B(t), +\infty, \mathscr{B}_t, P_x^B]$ be the Brownian motion on R^2 and let $\varphi(\alpha_1, \alpha_2 : t : x)$ be the characteristic function of $B(t)$ starting from x, which is equal to $\exp\{i(\alpha_1 x_1+\alpha_2 x_2)-(\alpha_1^2+\alpha_2^2)t/2\}$. Observing the formula in Lemmas 1.1 and 1.2, we have the following:

$$\lim_{c\to\infty} \varphi^c(\alpha_1, \alpha_2 : t : x)=\varphi(\alpha_1, \alpha_2 : t : x). \tag{2.7}$$

Then (2.7) means that the distribution of $x(t)$ converges to the distribution of $B(t)$. Moreover we can show the following theorem.

Theorem 2.1. *The finite dimensional distribution of* $[x(t), P_x]$ *converges to the corresponding finite dimensional distribution of* $[B(t), P_x^B]$.

Remark. It is shown in [4] that

$$P_x\{x(t)\in\Gamma\}=\int_\Gamma g(t, x, y)dy \qquad (\Gamma\in\mathscr{B}(R^2)), \tag{2.8}$$

where

$$g(t, x, y)=\frac{e^{-c^2t}}{2\pi ct}\left[\delta(|x-y|-ct)+\frac{c\exp\{c^2t\sqrt{1-(|x-y|^2/c^2t)}\}}{\sqrt{1-(|x-y|^2/c^2t^2}}\right] \tag{2.9}$$

$$\sim\frac{1}{2\pi ct}\exp\{-(|x-y|^2/2t)\}(1+0(t^{-2})) \qquad (t\to\infty).$$

3. Weak convergence. Let C be the space of all continuous functions and $\mathscr{B}(C)$ be the Borel field generated by cylindrical sets in C.

Since both stochastic processes $x(\cdot)$ and $B(\cdot)$ are continuous in t, therefore they determine probability measures μ_x^c and μ_x on $(C, \mathscr{B}(C))$, respectively. Under these notations we get the following theorem.

Theorem 3.1. *The transport process with speed c ($c>0$) converges weakly to the two-dimensional Brownian motion as the speed c increases to infinity, that is, the sequence of the measures μ_x^c converges weakly to μ_x as $c\to\infty$.*

To prove the theorem, we prepare a lemma.

Lemma 3.1. $E_x[|x(t)-x(s)|^4]\le K|t-s|^2$, *where K is a constant number independent of c.*

Proof. Expanding both sides of the equality (i) in Lemma 1.2 in the Taylor series and comparing the coefficients of α_1^4, $\alpha_1^2\alpha_2^2$, and α_2^4, we get

$$
\begin{aligned}
(3.1) \qquad E_0[|x(t)|^4] &= E_0[|x_1(t)|^4] + 2E_0[x_1(t)x_2(t)|^2] + E_0[|x_2(t)|^4] \\
&= 4!\, e^{-c^2 t} \left\{ \frac{4c^4t^4}{21} \sum_{n=0}^{\infty} \frac{(c^2t)^{2n}}{2(2n)!\,(n+1)(n+2)} \right. \\
&\quad \left. + \left(\frac{1}{c^4} + \frac{5}{12}t^2\right) \sinh(c^2t) - \frac{t}{c^2} \cosh(c^2t) \right\} \\
&\leq 68t^2.
\end{aligned}
$$

From this, we can easily prove the lemma.

Combinig Theorem 2.1 and Lemma 3.1, we complete the proof of Theorem 3.1. (cf. [2, Chap. IX]).

References

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[5] Watson, G. N.: A Treatise on the Theory of Bessel Functions. Cambridge Univ. Press (1922).