139. A Note on Inverse Images of Closed Mappings

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This paper is concerned with three results pertaining to the following problem. Given a mapping f in class $\mathcal{C}$ with the range of f in class $\mathcal{D}$, when will the domain of f be in class $\mathcal{E}$? In case f is a closed continuous mapping onto a paracompact Hausdorff space, S. Hanai [2, Theorem 5, p. 302] has given necessary and sufficient conditions for the domain of f to be normal. In Theorem 1, we provide another proof for Hanai's result, and in Theorem 2, under the same hypothesis on f, we obtain analagous necessary and sufficient conditions for the domain of f to be collectionwise normal. Under fairly restrictive hypothesis, Theorem 4 gives necessary and sufficient conditions for the domain of a mapping to be an M-space in the sense of Morita [6, p. 379].

In what follows, all mappings are assumed to be continuous. As usual, if X is a set, $\mathcal{F}=\{F_\alpha : \alpha \in A\}$ a collection of subsets of X, and $S \subseteq X$, we let $\mathcal{F}|S=\{F_\alpha \cap S : \alpha \in A\}$.

Let f be a mapping from X to the T_1 space Y, C a closed subset of X, and m a cardinal number. f satisfies condition γ_m at C iff for any discrete collection $\{C_\alpha : \alpha \in A\}$ of $\leq m$ closed subsets of C, there exists a pairwise disjoint open collection $\{U_\alpha : \alpha \in A\}$ such that $C_\alpha \subseteq U_\alpha$ for all α. If f satisfies condition γ_m at C for all cardinals m, we say that f satisfies condition γ at C.

Lemma 1.1. *Let f be a closed mapping from the topological space X onto the T_1 regular space Y. Suppose that f satisfies condition γ_2 at $f^{-1}(y)$ for all y in Y. Then for any y in Y, closed subset C of $f^{-1}(y)$, and open set U containing C, there exists an opeu set V such that $C \subseteq V \subseteq \overline{V} \subseteq U$.*

Proof. Let the closed set C of $f^{-1}(y)$ be contained in the open set 0. Using condition γ_2, choose open sets W_1 and W_2 of X containing C and $(X-0) \cap f^{-1}(y)$ respectively. Then $K=(X-0)-W_2$ is closed and misses $f^{-1}(y)$. Hence by regularity of Y, choose an open set P of Y with $y \in P \subseteq \overline{P} \subseteq Y-f(K)$. If $V=W_1 \cap f^{-1}(P)$, then V is as desired.

Theorem 1. *Let f be a closed mapping from the topological space X onto the paracompact Hausdorff space Y. X is normal iff f satisfies condition γ_2 at $f^{-1}(y)$ for all y in Y.*

Proof. We show that every finite open cover of X has a locally finite closed shrink. Let $\mathcal{U}=\{U_i : i=1,2,\cdots,n\}$ be an open cover of X. For each y in Y, $f^{-1}(y)$ is normal and hence $\mathcal{U}|f^{-1}(y)$ has a $1-1$ closed shrink $\mathcal{F}_y=\{F_{y,i} : i=1,2,\cdots,n\}$ covering $f^{-1}(y)$. By Lemma 1.1, for each y in Y, we obtain open covers $\mathcal{V}_y=\{V_{y,i} : i=1,2,\cdots,n\}$ of $f^{-1}(y)$ such that $F_{y,i}\subseteq V_{y,i}\subseteq \overline{V_{y,i}}\subseteq U_i$ for all i and y in Y. Let $0_y=\cup\{V_{y,i} : i=1.2,\cdots,n\}$ and let $P_y=Y-f(X-0_y)$ for y in Y. Then $\mathcal{P}=\{P_y : y\in Y\}$ is an open cover of the paracompact space Y with a locally finite open refinement $\mathcal{W}=\{W_y : y\in Y\}$. Then $\mathcal{Z}=\{f^{-1}(W_y)\cap V_{y,i} : y\in Y, i=1,2,\cdots,n\}$ is a locally finite open cover of X. Also, since $\overline{f^{-1}(W_y)\cap V_{y,i}}\subseteq \overline{V_{y,i}}\subseteq U_i$, then $\overline{\mathcal{Z}}\prec\mathcal{U}$. Thus X is normal.

Theorem 2. *Let f be a closed mapping from the topological space X onto the paracompact Hausdorff space Y. X is collectionwise normal iff f satisfies condition γ at $f^{-1}(y)$ for all y in Y.*

Proof. Since necessity is clear, we only prove sufficiency. Since f satisfies condition γ_2 at $f^{-1}(y)$ for all y in Y, X is normal by Theorem 1. By a result of Dowker [1, Lemma 1, p. 308], to show that X is collectionwise normal, it suffices to show that X is normal and that for any closed set C of X and locally finite relatively open cover $\mathcal{U}$ of C, there exists a locally finite open cover $\mathcal{V}$ of C such that $\mathcal{V}|C\prec\mathcal{U}$. Thus let $\mathcal{U}=\{U_\alpha : \alpha\in A\}$ be a locally finite relatively open cover of the closed subset C of X. Let $\mathcal{W}=\{W_\alpha : \alpha\in A\}$ be an open collection in X such that $U_\alpha=W_\alpha\cap C$ for α in A. Let $Y_C=\{y : y \text{ in } Y \text{ and } f^{-1}(y)\cap C\neq\phi\}$. Then for each y in Y_C, by arguing as in Dowker's Lemma 1 above with respect to the locally finite relatively open cover $\mathcal{W}|(f^{-1}(y)\cap C)$ of $f^{-1}(y)\cap C$ for which condition γ holds, we obtain locally finite open collections $\mathcal{Z}_y=\{Z_{y,\alpha} : \alpha\in A\}$ covering $f^{-1}(y)\cap C$ with $Z_{y,\alpha}\subseteq W_\alpha$ for all α in A. For each y in Y_C, let $0_y=\cup\mathcal{Z}_y\cup(X-C)$. Note that if y' is in Y and $f^{-1}(y')\subseteq 0_y$, then $f^{-1}(y')\cap C$ is covered by $\mathcal{Z}_y$. Clearly $(0_y)_0=f^{-1}(Y-f(X-0_y))$ contains $f^{-1}(y)$ for all y in Y_C and thus

$$\{f[(0_y)_0] : y\in Y_C\}\cup(Y-f(C))$$

is an open cover of Y. By paracompactness of Y, obtain a locally finite open collection $\mathcal{P}=\{P_y : y\in Y_C\}$ in Y covering $f(C)$ such that $P_y\subseteq f[(0_y)_0]$ for y in Y_C. Consider now

$$\mathcal{V}=\{V_{y,\alpha}=f^{-1}(P_y)\cap Z_{y,\alpha} : y\in Y_C, \alpha\in A\}.$$

$\mathcal{V}$ is a locally finite open collection and clearly

$$V_{y,\alpha}\cap C=f^{-1}(P_y)\cap Z_{y,\alpha}\cap C\subseteq Z_{y,\alpha}\cap C\subseteq W_\alpha\cap C\subseteq U_\alpha$$

for y in Y_C and α in A. Thus $\mathcal{V}|C$ refines $\mathcal{U}$. It remains to show that $\mathcal{V}$ covers C. But if p is in C, then since $\mathcal{P}$ covers $f(C)$, $f^{-1}(f(p))\subseteq f^{-1}(P_y)$ for some y in Y_C. Hence $f^{-1}(f(p))\subseteq(0_y)_0\subseteq 0_y$. But as we have noted earlier in the proof, $p\in f^{-1}(f(p))\cap C\subseteq\cup\mathcal{Z}_y$ and hence $p\in Z_{y,\alpha}$ for some α. Thus p is in $f^{-1}(P_y)\cap Z_{y,\alpha}=V_{y,\alpha}$ and $\mathcal{V}$ covers

C. Thus X is collectionwise normal.

Corollary 1.2. *Let f be a closed mapping from the topological space X onto the paracompact Hausdorff space Y. Then X is collectionwise normal iff $f^{-1}(y)$ is collectionwise normal and Bdry $f^{-1}(y)$ satisfies condition γ for all y in Y.*

Proof. Necessity is clear. It is an easy consequence of Theorem 1 that X is normal. A routine argument shows that $f^{-1}(y)$ satisfies condition γ for all y in Y. Thus by Theorem 2, X is collectionwise normal.

Corollary 1.3. *Every normal M-space is collectionwise normal.*

Proof. Let X be a normal M-space and f a closed mapping from X onto a metric space Y, where $f^{-1}(y)$ is countably compact for all y in Y. The existence of such a map is guaranteed by Morita [6, Theorem 6.1, p. 379]. Since every discrete collection in $f^{-1}(y)$ is normal, $f^{-1}(y)$ satisfies condition γ for all y in Y. Thus X is collectionwise normal.

We now turn to the development of some mapping theorems for M-spaces. For the definition of M-space (M^*-space) see [6, p. 379] ([3, p. 752]). It is easy to see that these definitions are equivalent to the following: There exists a normal sequence $\{\mathcal{U}_i : i=1, 2, \cdots\}$ of open coverings (a sequence $\{\mathcal{F}_i : i=1, 2, \cdots\}$ of locally finite closed coverings) of X satisfying the condition below:

(*) If $\{x_i\}$ is a sequence in X with $x_i \in \mathrm{St}(x, \mathcal{U}_i)$ ($x_i \in \mathrm{St}(x, \mathcal{F}_i)$) for every i and some x in X, then $\{x_i\}$ has a cluster point.

We say that a collection $\{U_\alpha : \alpha \in A\}$ of open sets is a base at the subset S of the topological space X iff for any open neighborhood V of S there exists U_α such that $S \subseteq U_\alpha \subseteq V$. X is first countable at S iff X has a countable base at S. A space is countably compact if each of its countable open covers has a finite subcover.

It is well known that if X is compact Hausdorff, then X is first countable at its point p iff p is a G_δ point of X. Generalizing this we have

Lemma 1.4. *Let X be a M^*-space and C a countably compact subspace. If the closed neighborhoods of C are a base at C and if C is a G_δ set, then X is first countable at C.*

Proof. Let $\{\mathcal{F}_i\}$ be a sequence of locally finite closed covers of X with $\mathcal{F}_{i+1} \prec \mathcal{F}_i$ for all i and $\{\mathcal{F}_i\}$ satisfying (*). Then as in Ishii [3, Lemma 2.2, p. 752], we see that $\{\mathrm{St}(C, \mathcal{F}_i) : i=1, 2, \cdots\}$ has the property that if $\{x_i\}$ is a sequence in X with $x_i \in \mathrm{St}(C, \mathcal{F}_i)$ for all i, then $\{x_i\}$ has a cluster point in X. Since $\mathcal{F}_i$ is locally finite, for each i we have $C \subseteq [\mathrm{St}(C, \mathcal{F}_i)]^0$. Using the additional facts that C is a G_δ with its closed neighborhoods as a base, we can construct a sequence $\{V_i\}$ of open neighborhoods of C such that $\overline{V_{i+1}} \subseteq V_i$ for all $i, \bigcap \{V_i : i$

$=1, 2, \cdots\}=C$ and if $\{x_i\}$ is a sequence with $x_i \in V_i$ for all i, then $\{x_i\}$ has a cluster point. Clearly $\{V_i\}$ is a local base at C and X is first countable at C.

Corollary 1.5. *If X is T_1 regular and a M^*-space, then a compact subset C is a G_δ set iff X is first countable at C.*

It is easy to see that if $\{U_i\}$ is a localy base for the countably compact subset C of the topological space X, and if $U_{i+1} \subseteq U_i$ for all i, then any sequence $\{x_i\}$ with x_i in U_i for all i has a cluster point in C.

Theorem 3. *Let f be a closed mapping from the normal T_1 space X onto the countably compact space Y. Let X be first countable at each of its closed countably compact subsets. Then X is an M-space iff $f^{-1}(y)$ is an M-space for each y in Y.*

Proof. Necessity is clear so we prove sufficiency. Thus assume that $f^{-1}(y)$ is an M-space for all y in Y. Since X is normal, in order to see that it is an M-space, it suffices to exhibit a sequence $\{U_i : i =1, 2, \cdots\}$ of locally finite open covers satisfying the condition $(*)$. Slightly modifying the argument of [7, Lemma 1, p. 10] in the obvious way, we have that Bdry $f^{-1}(y)$ is countably compact for all y in Y. It is easy to see from the closedness of f and the countable compactness of Y that $K= \cup\{\text{Bdry } f^{-1}(y) : y \in Y\}$ is closed and countably compact. Choose a base $\{V_i\}$ at K of open sets with $\overline{V_{i+1}} \subseteq V_i$ for all i. Also for each y in Y, since $f^{-1}(y)$ is an M-space, we can choose sequences $\{\mathcal{U}_{y,i} : i=1, 2, \cdots\}$ of relatively open locally finite covers of $f^{-1}(y)$ which satisfy condition $(*)$. For each i let

$$\mathcal{W}_i = \cup\{\mathcal{U}_{y,i} \mid X-\overline{V_{i+1}} : y \in Y\} \cup \{V_i\}.$$

Then clearly for each i, $\mathcal{W}_i$ is a locally finite open cover of X. We show that $\{\mathcal{W}_i\}$ satisfies condition $(*)$. To do so, suppose that $\{x_i\}$ is a sequence in X with $x_i \in \text{St}(x, \mathcal{W}_i)$ for each i and some x. If $x \in K$, then $x_i \in V_i$ for all i and hence $\{x_i\}$ has a cluster point in K. If x is not in K, then there exists y in Y with $x \in [f^{-1}(y)]^0$. Since $\cap\{V_i : i =1, 2, \cdots\}=K$, there exists $n(x)$ such that if $j \geq n(x)$, then x is not in V_j. Thus we have that $x_j \in \text{St}(x, \mathcal{U}_{y,j})$ if $j \geq n(x)$ and since $\{\mathcal{U}_{y,i} : i\ 1, 2, \cdots\}$ satisfies condition $(*)$, $\{x_i\}$ has a cluster point. Hence X is an M-space.

By a similar argument, Theorem 3 still holds if "M-space" is replaced everywhere by "M^*-space".

Theorem 4. *Let f be a closed mapping from the normal T_1 space X onto the paracompact, locally compact space Y. Let every compact set in X be a G_δ. Then X is a paracompact M-space iff $f^{-1}(y)$ is a paracompact M-space for all y in Y and X is first countable at each of its compact sets.*

Proof. *Necessity.* Let X be a paracompact M-space. Clearly, for all y in Y, $f^{-1}(y)$ as a closed subset of X is a paracompact M-space. Since X is an M^*-space and each of its compact subsets is a G_δ, by Corollary 1.5 X is first countable at each of its compact sets.

Sufficiency. For all y in Y, assume that $f^{-1}(y)$ is a paracompact M-space and assume that X is first countable at each of its compact sets. It is easy to see from [7, Lemma 1, p. 10] that Bdry $f^{-1}(y)$ is countably compact for each y in Y. Thus since Bdry $f^{-1}(y)$ is paracompact and T_1, it is compact for all y in Y. Applying [3, Corollary 2.4, p. 304], X is paracompact. Let $\mathcal{V}=\{V_\alpha : \alpha \in A\}$ be a locally finite collection of compact subsets of Y which covers Y. Since X is first countable at each of its compact and hence countably compact sets, we can apply Theorem 3 to $f^{-1}(V_\alpha)$ and $f|f^{-1}(V_\alpha)$ to conclude that $f^{-1}(V_\alpha)$ is an M-space for all α in A. Then $\{f^{-1}(V_\alpha) : \alpha \in A\}$ is locally finite and a closed cover of X by M-spaces. As Morita has noted, it is an easy consequence of [4, Theorem 1.1, p. 757] that X is an M-space.

Corollary 1.6. *Let X be a normal T_1 space with point-countable base. Let f be a closed mapping from X onto the paracompact, locally compact space Y. Then X is a paracompact M-space iff $f^{-1}(y)$ is a paracompact M-space for all y in Y.*

Proof. Necessity is trivial. For sufficiency, merely note that by [5, Theorem 1, p. 855], X is first countable at each of its compact sets. Thus by Theorem 4, X is a paracompact M-space.

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