131. A Note on Semi-prime Modules. II

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The main purpose of this note is to prove the following two theorems :[1]

Theorem A. *Let R be a semi-prime Goldie ring, let Q be the right quotient ring of R, and let R_i $(i=1, \cdots, t)$ be the minimal annihilator ideals*[2] *of R. Let M be a semi-prime R-module, let M_i be the subisomorphism classes of basic submodules*[3] *of M which corresponds to R_i and let J_i be a uniform right ideal contained in R_i $(i=1, \cdots, t)$. Then*

(i) *There exists an element $x_i \in M_i$ such that $I_i = \mathrm{Hom}_R(x_iJ_i, x_iJ_i)$ is a right Ore domain. The ring D_i $\mathrm{Hom}_R(x_iJ_iQ, x_iJ_iQ)$ is the right quotient division ring of I_i $(i=1, \cdots, t)$.*

(ii) *The ring $I = \mathrm{Hom}_R(N, N)$ is isomorphic onto $I_1 \oplus \cdots \oplus I_t$, where $N = x_1J_1 \oplus \cdots \oplus x_tJ_t$.*

(iii) *The ring $D = \mathrm{Hom}_R(NQ, NQ)$ is the right quotient ring of I and is isomorphic onto $D_1 \oplus \cdots \oplus D_t$.*

Theorem B. *Let R be a Goldie ring. If M is a semi-prime R-module, then M contains N, which is a direct sum of uniform submodules and R is contained in a semi-prime ring B such that the pair (B, N) has the double centralizer property. The submodule N may be chosen to be of the form $x_1J_1 \oplus \cdots \oplus x_tJ_t$, where $x_i \in M_i$ and J_i is a uniform right ideal in R_i $(i=1, \cdots, t)$.*

1. Proof of Theorem A. Lemma 1. *Let M be a semi-prime R-module and let Q be the right quotient ring of R. Then the injective envelope $\tilde{M}$ of M is MQ.*

Proof. Let $x = mc^{-1}$ be a non-zero element of MQ. Then $xc = m \in M \cap xR$, which implies that MQ is an essential extension of M. Suppose that M' is an essential extension of M, then $M'^{\blacktriangle} = 0$ and M' is faithful. Hence, by Proposition 1 in [7], M' is also semi-prime. By Proposition 4.1 in [3], we have $MQ = M'Q \supseteq M'$, which proves the lemma.

Since MQ is the injective envelope of M and $M^{\blacktriangle} = 0$, we may

1) Throughout this paper, definitions and notations are used in the same sense as in [7]. R will denote a right Goldie ring and all R-modules will mean faithful right R-modules.

2) Cf. [5. p. 215].

3) Cf. [7. Theorem 7].

assume[4] that $\mathrm{Hom}_R(M, M) \subseteq \mathrm{Hom}_R(MQ, MQ)$.

Lemma 2. *Let U and V be uniform submodules of a semi-prime R-module M. Then*

(i) $\mathrm{Hom}_R(U, U)$ *is an integral domain and* $\mathrm{Hom}_R(UQ, UQ)$ *is a division ring containing* $\mathrm{Hom}_R(U, U)$.

(ii) *If U and V are not connected, then* $\mathrm{Hom}_R(U, V)=0$.

(iii) *If U, V are basic submodules such that $U \sim V$, then* $\mathrm{Hom}_R(U, V)$ *is subisomorphic to* $\mathrm{Hom}_R(U, U)$ *as Z-modules.*

Proof. (i) follows at once by using the similar method as in Theorem 4.3 in [2]. (ii): Suppose that $\mathrm{Hom}_R(U, V) \neq 0$. Let B be a basic submodule of U and let f be a non-zero element in $\mathrm{Hom}_R(U, V)$. By Lemma 5.4 in [6], f is an isomorphism and thus $f(B) \cong B$. Hence we have $U \sim V$, which is a contradiction. (iii): By the assumption, there exists an isomorphism θ of U into V. If we define θ^* by $\theta^*(f)=\theta \cdot f$ for all $f \in \mathrm{Hom}_R(U, U)$, then it follows directly that θ^* is a Z-isomorphism of $\mathrm{Hom}_R(U, U)$ into $\mathrm{Hom}_R(U, V)$. Likewise an isomorphism $\phi: V \to U$ induces a Z-isomorphism ϕ^* of $\mathrm{Hom}_R(U, V)$ into $\mathrm{Hom}_R(U, U)$.

By the above two lemmas and the similar method in the proof of Theorem 4.4 in [2] we have

Proposition 3. *A semi-prime R-module M is uniform if and only if* $\mathrm{Hom}_R(\tilde{M}, \tilde{M})$ *is a division ring, where $\tilde{M}$ is the injective envelope of M.*

Lemma 4. *Let I and J be ideals of a ring A, and let $Q(I)$ and $Q(J)$ be the right quotient rings of I and J respectively. If $I+J$ is a direct sum, then $Q(I) \oplus Q(J)$ is the right quotient ring of $I \oplus J$.*

Proof. Let c, d be regular elements of I, J respectively. Then it follows at once that $c+d$ is regular in $I \oplus J$ and $(c+d)^{-1}=c^{-1}+d^{-1}$ in $Q(I) \oplus Q(J)$. Let $x=ac^{-1}+bd^{-1}$ be an element of $Q(I) \oplus Q(J)$. Then we have easily $x=(a+b)(c+d)^{-1}$, completing the proof.

Lemma 5. *Let J_i be a uniform right ideal contained in R_i $(i=1, \cdots, t)$. Then*

(i) *There exists an element $x_i \in M_i$ such that $x_iJ_i \cong J_i$ $(i=1, \cdots, t)$.*

(ii) *For each element $y \in M_k$, $yJ_i=0$ $(i \neq k)$.*

Proof. Since M is faithful, we have $MJ_i \neq 0$. Hence there exists an element m of M such that $mJ_i \neq 0$. By Theorem 2.4 in [2], we have $mJ_i \cong J_i$ and hence $mJ_i \subseteq M_i$. Since R is semi-prime, $(mJ_i)J_i \neq 0$. Thus there exists an element $x_i \in mJ_i \subseteq M_i$ such that $x_iJ_i \neq 0$. Again, by Theorem 2.4 in [2] we have $x_iJ_i \cong J_i$, which gives (i). To prove (ii), we suppose that y is any element of M_k. If $yJ_i \neq 0$ $(i \neq k)$, then, by

4) Cf. [2; p. 1047].

Theorem 2.4 in [2], $yJ_i \cong J_i$. Hence we have $yJ_i \subseteq M_i$. This contradicts the fact that $M_i \oplus M_k$ is a direct sum.

Proof of Theorem A. (i): By the similar methods as in Theorem 4 of [4] and [4, p. 607], it follows at once that $\mathrm{Hom}_R(J_i, J_i)$ is a right Ore domain with the right quotient division ring $\mathrm{Hom}_R(J_iQ, J_iQ)$ $(i=1, \cdots, t)$. By Lemma 5, we have $x_iJ_i \cong J_i$ for some $x_i \in M_i$ and $x_iJ_iQ \cong J_iQ$. Consequently, $I_i \cong \mathrm{Hom}_R(J_i, J_i)$ and $D_i \cong \mathrm{Hom}_R(J_iQ, J_iQ)$. Hence I_i is a right Ore domain with the right quotient division ring D_i $(i=1, \cdots, t)$. (ii): Let f be an element of $\mathrm{Hom}_R(N, N)$ and put $f_i = f \mid x_iJ_i$. Then we shall prove that $f_i(x_iJ_i) \subseteq x_iJ_i$. Let $a \in x_iJ_i$ and write

$$f_i(a) = a_1 + \cdots + a_t; \qquad (a_j \in x_jJ_j).$$

As a runs over x_iJ_i, the map $\theta_k : a \to a_k$ is a homomorphism of x_iJ_i into x_kJ_k. By Lemma 2, $\theta_k = 0$ for each $k \neq i$. And thus $f_i(x_iJ_i) \subseteq x_iJ_i$. As is easily shown, the map

$$f \to f_i + \cdots + f_t$$

is a ring-isomorphism of I onto $I_1 \oplus \cdots \oplus I_t$.

(iii) is immediately proved by Lemma 4 and the fact that J_iQ are mutually non-isomorphic minimal right ideals of Q.

Remark. This theorem is a generalization of Theorem 4.6 in [2].

Corollary. *The ring Q is isomorphic to $(D_1)_{n_1} \oplus \cdots \oplus (D_t)_{n_t}$, where $D_i = \mathrm{Hom}_R(x_iJ_iQ, x_iJ_iQ)$ and $(D_i)_{n_i}$ is a total matrix ring over D_i $(i=1, \cdots, t)$.*

2. Proof of Theorem B. Let x_iJ_i, I_i, D_i, I, D, and N be as in Theorem A. Now we shall show that N is a faithful R-module. Suppose that $Na=0$ for some element a of R, then we have $x_iJ_ia=0$ and hence $J_ia=0$ $(i=1, \cdots, t)$. Since J_i is representive for uniform right ideals, a annihilates for all uniform right ideals. And thus we have $R_0a=0$, where R_0 is the sum of all uniform right ideals in R. By [5, p. 207] we have $a=0$, as desired.

Since Q satisfies the maximum condition for right ideals, NQ is a finitely generated right Q-module. Then by Theorems 58.14 and 59.7 in [1], NQ is a finitely generated injective left D-module. Since D is the quotient ring of I, we have, by Corollary 4.2 in [2], NQ is an injective left I-module. From Theorem 59.6 in [1], the pair (Q, NQ) has the double centralizer property, i.e., $Q \cong \mathrm{Hom}_D(NQ, NQ)$. By Proposition 4.1 in [2], $Q \cong \mathrm{Hom}_I(NQ, NQ)$. Since NQ is an injective left I-module, each $\alpha \in \mathrm{Hom}_I(N, N)$ has an extension $\alpha^* \in \mathrm{Hom}_I(NQ, NQ)$. However, since N is a faithful R-module and $Q \cong \mathrm{Hom}_I(NQ, NQ)$, α^* is a unique extension of α. We may, therefore, write

$$R \subseteq B \subseteq Q,$$

where $B = \mathrm{Hom}_I(N, N)$. Therefore, Q is the right quotient ring of B, and by [5], B is a semi-prime Goldie ring. It is easily proved that the pair (B, N) has the double centralizer property.

Corollary. *Every semi-prime Goldie ring R is contained in a semi-prime ring B, which has the same quotient ring as R and satisfies the following properties:*

(i) *B is the ring of endomorphisms of the left module N over a direct sum of integral domains.*

(ii) *the pair (B, N) has the double centralizer property.*

Remark. Theorem B and Corollary are generalizations of Theorems 4.9 and 4.10 in [2] respectively.

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