183. *Approximation of Semigroups of Operators on Fréchet Spaces*

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1. Introduction. Let $\mathfrak{X}$ be a Fréchet space (cf. Treves [12], Chap. 10, pp. 85-94), and let $\mathcal{L}(\mathfrak{X})$ be the space of continuous linear transformations of $\mathfrak{X}$ into itself. Let $\{T(t), t \in R^+\}$, $T(t): R^+ \to \mathcal{L}(\mathfrak{X})$, be a one-parameter family of continuous operators. The family $\{T(t), t \in R^+\}$ is called a *semigroup of operators* if

(1) $$T(s+t)=T(s)\,T(t),\ s,\ t \in R^+,\ T(0)=I.$$

The *infinitesimal generator* of the semigroup $T(t)$ is defined as

(2) $$A=s-\lim_{h\to 0}(T(h)-I)/h,$$

and $\mathcal{D}(A)$ is the set of all $f \in \mathfrak{X}$ for which the above limit exists. The *resolvent operator* is difined as the abstract Laplace transform of $T(t)$, that is

(3) $$R(\lambda\,;A)f=\int_0^\infty e^{-\lambda t}T(t)f\,dt,\ f \in \mathfrak{X}.$$

The theory of semigroups on Fréchet spaces, which is a generalization of the theory of semigroups on Banach spaces, has been developed by Komatsu [5], Mate [6], Miyadera [7], Schwartz [10], and Yosida [14]. The study of the approximation of semigroups on Banach spaces was initiated by Trotter [13] (cf. also Kato [4]). We refer to Hasegawa [2], Ôharu [9] for other results on the aproximation of semigroups on Banach spaces, and to Ôharu [8] and Yosida [14] for the generalization of Trotter's results to locally convex topological vector spaces.

In this paper we state some results on the approximation of semigroups on Fréchet spaces, and consider as a concrete example, the approximation of a semigroup on the Fréchet space of infinitely differentiable functions, utilizing Chlodovsky's [1] generalizations of Bernstein polynomials on an infinite interval. The proofs subsidiary results will be given elsewhere. We remark that Seidman [11] independently obtained some of our results, following the methodology of Yosida.

2. Convergence of semigroups on Fréchet spaces. *Approximation theorems.* In this section we consider a sequence of Fréchet spaces $\{\mathfrak{X}_n\}$, $\mathfrak{X}_1 \subset \mathfrak{X}_2 \subset \cdots \subset \mathfrak{X}_n \subset \mathfrak{X}_{n+1} \subset \cdots$, and a countable family of seminorms $\{p_\gamma, \gamma \in \Gamma\}$ which topologizes $\mathfrak{X}$ and $\mathfrak{X}_n$, $n=1, 2, \cdots$. $\{\mathfrak{X}_n\}$ is called a *sequence of Fréchet spaces approximating* $\mathfrak{X}$ if there exists a sequence of linear maps $\{P_n\}$, $P_n : \mathfrak{X} \to \mathfrak{X}_n$, such that for each $f \in \mathfrak{X}$ and $\gamma \in \Gamma$

$$p_\gamma(P_n f) \leq P_\gamma(f), \lim_{n \to \infty} p_\gamma(f - P_n f) = 0. \tag{4}$$

Given a sequence of operators $\{A_n\}$, $A_n : \mathfrak{X}_n \to \mathfrak{X}_n$, $n=1, 2, \cdots$, and an operator $A : \mathfrak{X} \to \mathfrak{X}$, by $s\text{-}\lim_{n\to\infty} A_n = A$ we shall understand that for each $f \in \mathfrak{X}$ and each $\gamma \in \Gamma$, $\lim_{n\to\infty} p_\gamma(A_n P_n f - P_n A f) = 0$.

We now state the two basic approximation theorems.

Theorem 1. *Let* $\{T_n(t), t \in R^+\}$, $T_n(t) : \mathfrak{X}_n \to \mathfrak{X}_n$, $n=1, 2, \cdots$, *be a sequence of semigroups of operators of class* (C_0) *with associated resolvent operators* $\{R_n(\lambda)\}$ *satisfying the following conditions: for each* $\gamma \in \Gamma$ *and* $f_n \in \mathfrak{X}_n$

$$p_\gamma(T_n(t) f_n) \leq M_\gamma p_\gamma(f_n), \tag{5}$$

M_γ *being independent of t and n,*

$$p_\gamma(\lambda^m R_n^m(\lambda) f_n) \leq M_\gamma p_\gamma(f_n), \ m=1, 2, \cdots, \tag{6}$$

$$\lim_{\lambda \to \infty} p_\gamma[(\lambda R_n(\lambda) - I) f_n] = 0, \tag{7}$$

$$R_n(\lambda) - R_n(\mu) = (\mu - \lambda) R_n(\lambda) R_n(\mu), \ \lambda, \mu > 0. \tag{8}$$

If there exists a resolvent operator $R(\lambda)$, *satisfying the conditions* (6)–(8); *and such that* $s\text{-}\lim_{n\to\infty} R_n(\lambda) = R(\lambda)$, *then* $s\text{-}\lim_{n\to\infty} T_n(t) = T(t)$, *where* $\{T(t), t \in R^+\}$ *is a semigroup of class* (C_0) *on* $\mathfrak{X}$ *with resolvent operator* $R(\lambda)$.

The stability condition (5) may be expressed in a more general form, and then the following theorem holds.

Theorem 2. *Let* $\{T_n(t), t \in R^+\}$, $T_n(t) : \mathfrak{X}_n \to \mathfrak{X}_n$, $n=1, 2, \cdots$, *be a sequence of semigroups of operators of class* (C_0) *with associated infinitesimal generators* $\{A_n\}$ *satisfying the following conditions:* (i) *for each* $\gamma \in \Gamma$ *and for each* $f_n \in \mathfrak{X}_n$, $p_\gamma(T_n(t) f_n) \leq M_\gamma e^{\sigma t} p_\gamma(f_n)$, *where* M_γ *and* σ *are independent of n and t,* (ii) $A = \lim_{n\to\infty} A_n$ *is densely defined,* (iii) *for some* $\lambda > \sigma$, $\mathcal{R}(\lambda I - A)$ *is dense in* $\mathfrak{X}$. *Then the closure of A is the infinitesimal generator of a semigroup* $T(t)$ *of class* (C_0), *and* $T(t) = s\text{-}\lim_{n\to\infty} T_n(t)$.

3. Approximation by discrete parameter semigroups. An operator on $\mathfrak{X}$, whose powers are uniformly locally bounded may be utilized to construct a semigroup depending upon a parameter varying in a discrete set. We state a lemma which gives a method of constructing a discrete parameter semigroup.

Lemma. *Let* $T : \mathfrak{X} \to \mathfrak{X}$, *be an operator such that for each* $\gamma \in \Gamma$ *and for each* $f \in \mathfrak{X}$, $p_\gamma(T^k f) \leq M_\gamma p_\gamma(f)$, $k=1, 2, \cdots$. *Then, for* $h > 0$,

$A=(T-I)/h$ is the infinitesimal generator of a semigroup

$$S(t)=\sum_{k=0}^{\infty}(tA)^k/k!=e^{-t/h}\sum_{k=0}^{\infty}\left(\frac{t}{h}T\right)^k/k!,$$

such that for each $\gamma\in\Gamma$ and $f\in\mathfrak{X}$, $p_\gamma(S(t)f)\leq M_\gamma p_\gamma(f)$.

The parameter may be allowed to vary in a suitable discrete set; this device is useful in approximation theory. We now state an approximation theorem which can be deduced from Theorem 2 and Lemma.

Theorem 3. *Let h_n be a sequence of positive numbers converging to zero, and let $\{T_n\}$ be a sequence of operators on $\mathfrak{X}_n$, such that for each $\gamma\in\Gamma$ and for each $f_n\in\mathfrak{X}_n$, $p_\gamma(T_n^kf_n)\leq M_\gamma e^{\sigma kh_n}p_\gamma(f_n)$, where M_γ and σ are independent of n and k. Let $A_n=(T_n-I)/h_n$. Suppose that* (i) *$A=\lim_{n\to\infty}A_n$ is densely defined,* (ii) *for some $\lambda>\sigma$, $\mathcal{R}(\lambda I-A)$ is dense in $\mathfrak{X}$. Then the closure of A is the infinitesimal generator of a semigroup $S(t)=\lim_{n\to\infty}T_n[t/h_n]$.*

4. Approximation of a semigroup of operators on a space of infinitely differentiable functions. Consider the Fréchet space $\mathfrak{X}$ of all functions $f(\xi)$, $\xi\in R$, having the following properties: (a) $\sup_{\xi\in(-b_n,b_n)}|f(\xi)|\leq M_f(b_n)$ where $b_n=o(n)$, $b_n>0$, and $\{b_n\}$ is a strictly monotonic increasing sequence, and $M_f(b_n)e^{-\alpha n/b_n}\to 0$ for each $\alpha>0$; (b) the family of seminorms $\{p_n(\cdot)\}$ is defined by $p_n(f)=\sup_{\xi\in(-b_n,b_n)}|f(\xi)|$, $n=1,2,\cdots$; (c) f is infinitely differentiable at each $\xi\in R$, and $P_n\left(\frac{d^m}{d\xi^m}f(\xi)\right)\leq Kp_n(f(\xi))$ where the constant K is independent of m and n. Consider the differential operator $D=d/d\xi$; then $D^m=d^m/d\xi^m$, $m=1,2,\cdots$. It follows that $S(t)=e^{-t}\sum_{k=0}^{\infty}(tD)^k/k!$ is a semigroup of operators on $\mathfrak{X}$, with infinitesimal generator $A=D-I$.

Let $\mathfrak{X}_n$ be the space generated by the set of polynomials of degree n, satisfying the conditions (a)-(c). Then $\mathfrak{X}_n\subset\mathfrak{X}$. Consider the operators $P_n:\mathfrak{X}\to\mathfrak{X}_n$ defined by

$$P_n(f(\xi))=\sum_{\nu=0}^{n}f\left(\frac{b_{n\nu}}{n}\right)\binom{n}{\nu}\left(\frac{\xi}{b_n}\right)^\nu\left(1-\frac{\xi}{b_n}\right)^{n-\nu}.$$

From the results of Chlodovsky [1], it follows that the operators P_n satisfy the condition (4). Consider now the difference operator

$$\Delta_nf(\xi)=(f(\xi)-f(\xi-h_n))/h_n,\ \text{for } 0<\xi<b_n,$$
$$=(f(\xi+h_n)-f(\xi))/h_n,\ \text{for } -b_n<\xi\leq 0,$$

where $0<h_n<b_n/n^2$. Then $S_n(t)=e^{-t}\sum_{k=0}^{\infty}(t\Delta_n)^k/k!$ is a semigroup of operators on $\mathfrak{X}_n$, with infinitesimal generator $A_n=\Delta_n-I$, $n=1,2,\cdots$. It can be proved that $S_n(t)\to S(t)$ strongly. This semigroup may be

regarded as a Fréchet space analogue of the semigroup of translations in the Banach space $C[0, \infty]$.

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