

## 179. A Metric Characterization of the Cartesean Decomposition in a $\star$ -Algebra

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1. Ky Fan and A. J. Hoffman [2] observed, among others :

If  $T$  is an  $n \times n$  matrix and if an  $n \times n$  matrix  $A$  satisfies  $A = \text{Re } T$ , then

$$(1) \quad \|T - A\|_* \leq \|T - H\|_*$$

for any hermitean  $n \times n$  matrix  $H$ , where  $\|C\|_*$  is a unitarily invariant norm of  $C$ .

Very recently, the theorem of Fan and Hoffman is generalized for an operator  $T$  belonging to a finite factor by Marie and Hisashi Choda [1] under the restriction that the norm is defined by

$$(2) \quad \|C\|_*^2 = \varphi(C^*C),$$

where  $\varphi$  is the trace of factor. But the norm the condition (2) is too restrictive so that the theorem of Fan and Hoffman is excluded.

In the present note, we shall give an abstract formulation which includes the both of the theorems of Fan-Hoffman and Choda. Through this formulation, we shall show that the self adjoint operator  $A$  in the Cartesean Decomposition is the nearest self adjoint operator to the given  $T$  in  $\star$ -algebra  $\mathfrak{A}$ , which will give a metric characterization of the Cartesean Decomposition in  $\star$ -algebra  $\mathfrak{A}$ .

We should like to express here our cordial thanks to Professor Masahiro Nakamura for his kind advice in the preparation of this paper.

2. Throughout this note, we shall assume that  $\mathfrak{A}$  be a linear space with an involution  $x \rightarrow x^*$  ([3])

$$I 1 \quad (\alpha x + \beta y)^* = \alpha^* x^* + \beta^* y^*,$$

$$I 2 \quad x^{**} = x,$$

where  $\alpha^*$  is the complex conjugate of  $\alpha$ . An element  $T$  of  $\mathfrak{A}$  will be called self adjoint or hermitean if  $T^* = T$ . It is easy to deduce that the set  $\mathfrak{A}^s$  of all hermitean members of  $\mathfrak{A}$  is a real linear subspace of  $\mathfrak{A}$ , whence  $\mathfrak{A}^s$  is convex. Let  $T$  be an element of  $\mathfrak{A}$ , then we have the cartesean decomposition of  $T$  by

$$(3) \quad T = \text{Re } T + i \text{Im } T,$$

where  $\text{Re } T$  and  $\text{Im } T$  are self adjoint which are defined by

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$$(4) \quad \operatorname{Re} T = \frac{1}{2}(T + T^*) \text{ and } \operatorname{Im} T = \frac{1}{2i}(T - T^*)$$

which are called the real part and imaginary part of  $T$  respectively. Now we shall assume further that  $\mathfrak{X}$  is a normed space and that the norm is adjoint preserving in the sense that

$$(5) \quad \|T^*\| = \|T\|.$$

This weakens the unitary invariance of norm in Fan and Hoffman's theorem. It is easy to show that involution  $*$  is continuous under (5). Consequently  $\mathfrak{X}^s$  is closed if (5) is satisfied.

The following formulation includes the theorems of Fan-Hoffman and Choda :

**Theorem 1.** *If  $\mathfrak{X}$  is a  $\star$ -linear normed space which satisfied (5), then*

$$\left\| T - \frac{T + T^*}{2} \right\| \leq \|T - H\|$$

*holds for any self adjoint operator  $H$  in  $\mathfrak{X}$ .*

The proof of the theorem is completely same as that of Fan and Hoffman.

$$\begin{aligned} \left\| T - \frac{T + T^*}{2} \right\| &= \left\| \frac{T - H}{2} + \frac{H - T^*}{2} \right\| \\ &\leq \frac{1}{2} \|T - H\| + \frac{1}{2} \|T^* - H\| \\ &= \|T - H\| \end{aligned}$$

3. We shall give a converse of Theorem 1 in the case of  $C^*$ -algebra.

**Theorem 2.** *Let  $T$  be an operator in a  $C^*$ -algebra with a faithful trace  $\varphi$ , which satisfies (1), then  $A = \operatorname{Re} T$ .*

**Proof.** Let  $T = A' + iB$  be the cartesean decomposition of  $T$ . By the definition of the norm and the assumption (1) we have the following inequality

$$0 \leq \varphi((T - A)^*(T - A)) \leq \varphi((T - H)^*(T - H))$$

for any self adjoint operator  $H$  in  $\mathfrak{X}$ .

Hence we have

$$0 \leq \varphi((A' - A - iB)(A' - A + iB)) \leq \varphi((A' - H - iB)(A' - H + iB)).$$

This means

$$0 \leq \varphi((A' - A)^2 + B^2) \leq \varphi((A' - H)^2 + B^2)$$

which is equivalent to

$$0 \leq \varphi((A' - A)^2) \leq \varphi((A' - H)^2).$$

Since  $H$  is any self adjoint operator in  $\mathfrak{A}$ , we put  $H=A'$ , so we get

$$(6) \quad \varphi((A'-A)^2)=0.$$

$(A'-A)^2$  is positive and  $\varphi$  is faithful, (6) implies

$$A=A'=\text{Re } T. \quad \text{q.e.d.}$$

During the preparation of this paper, Professor M. Nakamura kindly informed us in his private letter that Theorem 2 in the case of finite factors had been proved by M. Choda and H. Choda independently.

4. Now we shall give a generalization of Theorem 2.

**Theorem 3.** *If  $\mathfrak{A}$  is a  $\star$ -linear normed space which satisfies (5). Moreover if the norm is strictly convex, then the following two conditions are equivalent:*

- (i)  $A$  is a real part of  $T$ ,
- (ii)  $A$  self adjoint operator  $A$  satisfies (1).

**Proof.** We have only to prove that (ii) implies (i) since the converse is valid in Theorem 1. Put  $H=\frac{1}{2}(T+T^*)$  in (1). On the other hand by Theorem 1 we have

$$\|T-H\| \leq \|T-A\|$$

since we may take  $A$  instead of  $H$  in Theorem 1. Thus we get

$$\|T-H\| = \|T-A\|$$

by the assumption (1). However  $\mathfrak{A}^s$  is convex closed and our norm is strictly convex, there exists a unique  $A$  in  $\mathfrak{A}^s$  with minimum distance  $\|T-A\|$  so we can conclude

$$A=H=\text{Re } T.$$

So the proof is complete.

The strictly convexity of norm in Theorem 3 cannot be removed. We show

**Theorem 4.** *If  $\mathfrak{A}$  is a  $\star$ -linear normed space which satisfies (5) without the strictly convexity of norm, then the implication (ii) $\rightarrow$ (i) does not always hold.*

**Proof.** Let  $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$  be a matrix of  $3 \times 3$  on a 3-dimensional

Euclidean space  $E^3$ .

We define the norm of  $T$  as usual

$$\|T\| = \sup \|Tx\|$$

for every unit vector  $x$  in  $E^3$ . We have

$$\text{Re } T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \|T - \text{Re } T\| = \left\| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right\| = 1.$$

Let a self adjoint operator  $A' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ ,  $A' \neq \operatorname{Re} T$ .

But  $\|T - A'\| = \left\| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right\| = 1$ . Thus  $\|T - A'\| = \|T - \operatorname{Re} T\|$ .

This shows the failure of implication (ii)→(i). In this case the matrix norm  $\|T\|$  is not strictly convex. So we get the proof of Theorem 4.

### References

- [1] M. Choda and H. Choda: On the minimality of the polar decomposition in finite factors (to appear).
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- [3] C. E. Rickart: *General Theory of Banach Algebras*. The University Series in Higher Mathematics, Van Nostrand (1960).