208. On Definitions of Commutative Rings

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G. R. Blakley and S. Ôhashi, one of the present authors give some interesting axioms of commutative rings (see [1], [2]). In this note, we shall give axiom systems of commutative rings, and semirings.

We shall consider a system $\langle R, +, \cdot, -, 0, 1\rangle$, where R is a non-empty set, 0 and 1 are elements of R, $+$, and $\cdot$ are binary operations on R, and $-$ is a unary operation on R.

Theorem 1. *$\langle R, +, \cdot, -, 0, 1\rangle$ is a commutative ring, if it satisfies the following conditions:*

1) $r=0+r$,
2) $r\cdot 1=1\cdot r=r$,
3) $((-r)+r)\cdot a=0$,
4) $((c+(a\cdot y))+b)\cdot r=c\cdot r+(r\cdot b+a\cdot(y\cdot r))$.

As usual case, we omit the symbol $\cdot$ to write formulas. Therefore, ab means $a\cdot b$.

5) $(-r)+r=0$.

Proof.

$$\begin{aligned} 0&=((-r)+r)1 && \text{by 3)},\\ &=(-r)+r. && \text{by 2)}. \end{aligned}$$

6) $0a=0$.

Proof.

$$\begin{aligned} 0a&=((-r)+r)a && \text{by 5)},\\ &=0. && \text{by 3)}. \end{aligned}$$

7) $a+b=b+a$.

Proof.

$$\begin{aligned} a+b&=((0+(a1))+b)1 && \text{by 1), 2)},\\ &=01+((1b)+a(11)) && \text{by 4)},\\ &=0+(b+a) && \text{by 2), 6)},\\ &=b+a. && \text{by 1)}. \end{aligned}$$

8) $ab=ba$.

Proof.

$$\begin{aligned} ab&=(0+(00))+a)b && \text{by 1), 6)},\\ &=0b+(ba+0(0b)) && \text{by 4)},\\ &=0+(ba+0) && \text{by 6)},\\ &=0+(0+ba) && \text{by 7)},\\ &=ba. && \text{by 1)}. \end{aligned}$$

9) $(c+a)+b=c+(a+b)$.

Proof.

$$\begin{aligned}(c+a)+b&=((c+(a1))+b)1 && \text{by 2)},\\ &=c1+(1b+a(11)) && \text{by 4)},\\ &=c+(b+a) && \text{by 2)},\\ &=c+(a+b). && \text{by 7)}.\end{aligned}$$

10) $(ab)c=a(bc)$.

Proof.

$$\begin{aligned}(ab)c&=((0+(ab))+0)c && \text{by 1), 7)},\\ &=0c+(c0+a(bc)) && \text{by 4)},\\ &=0+(0+a(bc)) && \text{by 6), 8)},\\ &=a(bc). && \text{by 1)}.\end{aligned}$$

11) $(a+b)c=ac+bc$.

Proof.

$$\begin{aligned}(a+b)c&=((a+(00))+b)c && \text{by 1), 6), and 7)},\\ &=ac+(cb+a(0c)) && \text{by 4)},\\ &=ac+(cb+a0) && \text{by 6)},\\ &=ac+(0a+cb) && \text{by 7), 8)},\\ &=ac+cb && \text{by 1), 6)},\\ &=ac+bc. && \text{by 8)}.\end{aligned}$$

12) For given a, b, $a+x=b$ is solvable.

Proof. By 9), 7), 5), and 1), we have

$$a+((-a)+b)=(a+(-a))+b=0+b=b.$$

Therefore $x=(-a)+b$.

Hence the proof of Theorem 1 is complete.

Theorem 2. *An algebraic system $\langle R, +, \cdot, -, 0, 1\rangle$ is a semiring with zero and identity that the addition and multiplication are commutative, if it satisfies the following conditions:*

1) $0+r=r$,

2) $r1=1r=r$,

3) $0r=0$,

4) $((c+(ay))+b)r=cr+(rb+a(yr))$.

The proof is similar with the proof of Theorem 1. The proof is given by some steps.

5) $a+b=b+a$.

Proof.

$$\begin{aligned}a+b&=((0+(a1))+b)1=01+(1b+a(11))\\ &=0+(b+a)\\ &=b+a.\end{aligned}$$

6) $ab=ba$.

Proof.

$$ab=(0+(00))+a)b=0b+(ba+0(0b))$$

$$=0+(ba+0)=0+(0+ba)$$
$$=ba.$$

7) $(c+a)+b=c+(a+b)$.

Proof.

$$(c+a)+b=((c+(a1))+b)1$$
$$=c1+(1b+a(11))=c+(b+a)$$
$$=c+(a+b).$$

8) $(ab)c=a(bc)$.

Proof.

$$(ab)c=((0+(ab))+0)c=0c+(c0+a(bc))$$
$$=0+(0+a(bc))$$
$$=a(bc).$$

9) $(a+b)c=ac+bc$.

Proof.

$$(a+b)c=((a+(00))+b)c=ac+(cb+0(0c))$$
$$=ac+(cb+0)=ac+(0+cb)$$
$$=ac+cb$$
$$=ac+bc.$$

Therefore R is a semiring with zero and identity that the addition and the multiplication are commutative.

References

[1] G. R. Blakley: Four axioms for commutative rings. Notices of Amer. Math. Soc., **15**, p. 730 (1968).

[2] S. Ôhashi: On axiom systems of commutative rings. Proc. Japan Acad., **44**, 915-919 (1968).