74. *Boundedness of Solutions to Nonlinear Equations in Hilbert Space*

By Athanassios G. KARTSATOS[1]
Department of Mathematics, University of Athens, Greece

(Comm. by Zyoiti SUETUNA, M. J. A., May 12, 1969)

In what follows, by $H=(H, \langle , \rangle)$ we denote a complex Hilbert space, and by $B=B(H, H)$, the space of all bounded linear operators from H into H, associated with the strong operator topology. The only topology that we consider on H is the strong one.

Our aim in this paper is to give a boundedness theorem for the solutions of the differential equation

$$(*) \qquad \dot{x}=A(t)x+f(t, x),$$

where $x: I \to H$, $I=[t_0, +\infty)$, $t_0 \geq 0$, is a differentiable function on I with continuous first derivative,[2] $A: I \to B$ is a continuous function on I, and $f: I \times H \to H$ is also continuous on $I \times H$.

1. Theorem 1. *Consider* $(*)$ *under the following assumptions:*

(i) *there exists an operator valued function* $Q: I \to B$ *continuous and such that:*

$$(\mathrm{i}_1) \qquad \dot{Q}(t)+Q(t)A(t)+A^*(t)Q(t)=0,^{3)} \qquad t \in I,$$

and

$$(\mathrm{i}_2) \qquad |\langle Q(t)x, x\rangle| \geq g(\|x\|), \qquad (t, x) \in I \times H,$$

where $g: \boldsymbol{R}_+ \to \boldsymbol{R}_+=[0, +\infty)$ *is continuous and* $\limsup_{y \to +\infty} g(y)=+\infty$;

(ii) $\|x\| \cdot \|f(t, x)\| \leq p(t)g(\|x\|)$, *with* $p: I \to \boldsymbol{R}_+$ *continuous and such that*

$$\int_{t_0}^{\infty} p(t)\|Q(t)\| dt < +\infty;$$

then, if $x(t)$, $t \in I$, *is a solution of* $(*)$, *it is bounded, i.e. there exists a constant* $k>0$ *such that* $\|x(t)\| \leq k$ *for every* $t \in I$.

Proof. By differentiation of the function

$$(1) \qquad V(t)=\langle Q(t)x(t), x(t)\rangle,$$

we have

$$(2) \qquad \begin{aligned} \dot{V}(t) &= \langle \dot{Q}(t)x(t)+Q(t)\dot{x}(t), x(t)\rangle + \langle Q(t)x(t), \dot{x}(t)\rangle \\ &= \langle \dot{Q}(t)x(t)+Q(t)A(t)x(t)+Q(t)f(t, x(t)), x(x)\rangle \\ &\quad + \langle Q(t)x(t), A(t)x(t)+f(t, x(t))\rangle \\ &= \langle (\dot{Q}(t)+Q(t)A(t)+A^*(t)Q(t))x(t), x(t)\rangle \\ &\quad + \langle Q(t)f(t, x(t)), x(t)\rangle + \langle Q(t)x(t), f(t, x(t))\rangle \end{aligned}$$

and by integration from t_0 to t $(t_0 \leq t)$, we have

---

1) This research was supported in part by a NATO grant.

2) The existence of solutions on I is assumed without further mention.

3) $A^*(t)$ is the adjoint of the operator $A(t)$.

$$(3)\qquad V(t)=V(t_0)+\int_{t_0}^{t}[\langle Q(s)f(s, x(s)), x(s)\rangle+\langle Q(s)x(s), f(s, x(s))\rangle].$$

From (3) it follows that

$$(4)\qquad \begin{aligned} g(\|x(t)\|)\le |V(t)|\le |V(t_0)|+2\int_{t_0}^{t}\|Q(s)\|\cdot\|f(s, x(s))\|\cdot\|x(s)\|ds \\ \le |V(t_0)|+2\int_{t_0}^{t}p(s)\|Q(s)\|g(\|x(s)\|)ds, \end{aligned}$$

which, by a well known inequality, gives

$$(5)\qquad g(\|x(t)\|)\le |V(t_0)|\ \exp\left\{2\int_{t_0}^{t}p(s)\|Q(s)\|ds\right\},$$

and this proves the theorem.

Remark 1. If $g(\|x\|)=\lambda\|x\|^2$ (λ constant), then the condition (i_1) can be replaced by the following:

$$(i_3)\qquad \int_{t_0}^{\infty}\|\dot{Q}(t)+Q(t)A(t)+A^*(t)Q(t)\|dt<+\infty.$$

In fact, in this case from (2) we obtain

$$\lambda\|x(t)\|^2\le |V(t)|\le |V(t_0)|+\int_{t_0}^{t}\|\dot{Q}(s)+Q(s)A(s)+A^*(s)Q(s)\|\cdot\|x(s)\|^2ds$$
$$+2\lambda\int_{t_0}^{t}p(s)\|Q(s)\|\cdot\|x(s)\|^2ds,$$

and the proof follows as in Theorem 1.

Remark 2. Theorem 1 contains partially as a special case a result of Schaeffer in [1], who considered the linear equation

$$(**)\qquad \dot{x}=A(t)x$$

where $A(t)$ is an $n\times n$ (complex) matrix function, and x an $n\times n$ (complex) vector.

2. Theorem 2. *Suppose that in* (*) *the assumptions* (i) *are satisfied along with the following*:

$$(ii_a)\qquad \|x_1-x_2\|\cdot\|f(t, x_1)-f(t, x_2)\|\le\mu(t)g(x_1-x_2)$$

for every $(t, x_1, x_2)\in I\times H\times H$, *where* g *is as in* (i_2) *of Theorem* 1, *and* $\mu: I\to \boldsymbol{R}_+$ *is a continuous function such that*

$$\int_{t_0}^{\infty}\mu(t)\|Q(t)\|dt<+\infty;$$

then if (*) *has a bounded solution* $y(t)$, *every solution of* (*) *is bounded.*

Proof. Suppose that $x(t)$ is any solution of (*); then for the difference $x(t)-y(t)$ we have

$$(6)\qquad x(t)-y(t)=A(t)(x(t)-y(t))+(f(t, x(t))-f(t, y(t)));$$

by differentiation of the function

$$(7)\qquad V_0(t)=\langle Q(t)(x(t)-y(t)), x(t)-y(t)\rangle,$$

and proceeding as in Theorem 1, we finally find

$$(8)\qquad g(\|x(t)-y(t)\|)\le |V_0(t)|\le |V_0(t_0)|\exp\left\{2\int_{t_0}^{t}\mu(s)\|Q(s)\|\,ds\right\},$$

thus

$$(9)\qquad \|x(t)\|-\|y(t)\|\le\|x(t)-y(t)\|\le K\quad \text{for every } t\in I,$$

and for some positive constant K.

Obviously, since $y(t)$ is bounded, our assertion is true.

Reference

[1] A. J. Schaeffer: Boundedness of solutions to linear differential equations. Bull. Amer. Math. Soc., **74**, 508–511 (1968).