104. *Convergence of Transport Process to Diffusion*

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Let us consider a transported particle in the transprot medium G, which is bounded or unbounded domain of R^n. Suppose that it travels in a straight line and interacts with the medium with probability $k\Delta + o(\Delta)$ during time t and $t+\Delta$. The scattering distribution of velocity from $c\omega$ to $c\omega'$, $\omega' \in d\omega'$ at point $x \in G$ is assumed given by $\pi_x(d\omega')$. If the particle hits the boundary of G, then it dies. Under these assumptions, the position $X(t)$ and velocity $V(t)$ of the particle at time t together make up a Markov process $(X(t), V(t))$.

The purpose of this paper is to show that when $c\to\infty$, the process $X(t)$ converges to a diffusion under some additional assumptions (Assumptions I, II, and III).

The same result has been obtained in case of one-dimensional transport process by N. Ikeda, H. Nomoto [1] and M. Pinsky [3]; in case of two-dimensional isotropic one by A. S. Monin [2] and T. Watanabe [6]; in case of multi-dimensional isotropic one by S. Watanabe and T. Watanabe [5].

1. Let G be bounded or unbounded domain of n-dimensional Euclidian space R^n. Suppose that the boundary ∂G of G is smooth, if it exists. Let Ω be a bounded set in R^n. Let denote by S the product space of R^n and Ω, and by $\boldsymbol{C}_0(S)$ the Banach space of bounded continuous function on S vanishing at infinity and at boundary point (x, ω) such that $(\boldsymbol{n}_x, \omega)\leq 0$, where $\boldsymbol{n}_x$ is an inner normal vector at $x \in \partial G$. Let $\boldsymbol{T}_t^c$, $t\geq 0$, be the strongly continuous positive contraction semigroup on $\boldsymbol{C}_0(S)$ with infinitesimal generator $\boldsymbol{A}^c$ given by:

$$\boldsymbol{A}^c f(x,\omega) = c(\omega, \operatorname{grad} f) + k\int_\Omega [f(x,\upsilon) - f(x,\omega)]\,d\pi_x(\upsilon),$$

where $(\omega, \operatorname{grad} f) = \sum_{i=1}^n \omega_i \frac{\partial}{\partial x_i} f$, $\omega=(\omega_1, \cdots, \omega_n)$, and π_x $(x \in R^n)$ is a probability measure on Ω. We call this semigroup $\boldsymbol{T}_t^c$, $t\geq 0$, *the transport process with speed c.*

Now let $\boldsymbol{C}_0(G)$ be the Banach space of bounded continuous function on G vanishing near the boundary ∂G and at infinity, and $\boldsymbol{C}^3_{\mathcal{K}}(G)$ be the subspace of $\boldsymbol{C}_0(G)$ of function with compact support, whose thrice derivatives belong to $\boldsymbol{C}_0(G)$. Let $\boldsymbol{T}_t^D$, $t\geq 0$, be the strongly continuous positive contraction semigroup on $\boldsymbol{C}_0(G)$ of diffusion determined by:

$$DF(x)=\sum_{1\le i,j\le n} a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_i}F(x)+\sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}F(x).$$

We shall always assume the following:

(I) $c^2/k=d^2$ (constant)

(II) $D(C^3_{\mathcal{K}}(G))=\{U=DF : F\in C^3_{\mathcal{K}}(G)\}$ is dense in $C_0(G)$

(III) $s-\lim_{h\to 0}\int_\Omega \frac{[F(x+h\omega)-F(x)]}{h^2}d\pi_x(\omega)=\frac{1}{d^2}DF(x)$

for every $F\in C^3_{\mathcal{K}}(G)$.

Then we have

Theorem. *For every $F\in C_0(G)$,*

*$T^c_tF(x,\omega)\to T^D_tF(x)$ uniformly in (x,ω), as $c\to\infty$.**)

To prove the theorem, we prepare a following lemma mentioned in [4].

Lemma. *Let X and X_n, $n=1,2,\cdots$, be Banach spaces and $P_n: X\to X_n$ be linear maps such that $\|P_n\|\le 1$ and $\lim_{n\to\infty}\|P_nf\|=\|f\|$ for every $f\in X$. Let $T(t)$ and $T_n(t)$, $t\ge 0$, be strongly continuous positive contraction semigroups on X with infinitesimal generator A and on X_n with A_n, $n=1,2,\cdots$, respectively. Suppose that there exists a dense subset M of X such that $\|P_nAf-A_nP_nf\|\to 0$ for any $f\in M$ as $n\to\infty$ and $A(M)=\{g: g=Af, f\in M\}$ is dense in X. Then for every $f\in X$,*

$$\|P_nT(t)f-T_n(t)P_nf\|\to 0 \quad as \quad n\to\infty.$$

Proof of Theorem. For $F\in C_0(G)$, define $(P^cF)(x,\omega)=F\left(x+\frac{d}{\sqrt{k}}\omega\right)$ $\left(=0,\text{ if }\left(x+\frac{d}{\sqrt{k}}\omega\right)\notin G\right)$. Then it follows from Assumptions I, II, and III that, for $F\in C^3_{\mathcal{K}}(G)$,

$$\begin{aligned}A^c(P^cF)(x,\omega)&=c(\omega,\operatorname{grad}P^cF)+k\int_\Omega[(P^cF)(x,\upsilon)-(P^cF)(x,\omega)]d\pi_x(\upsilon)\\&=\left[c(\omega,\operatorname{grad}F)\left(x+\frac{d}{\sqrt{k}}\omega\right)-\sqrt{k}\frac{\left[F\left(x+\frac{d}{\sqrt{k}}\omega\right)-F(x)\right]}{(1/\sqrt{k})}\right]\\&\quad+\int_\Omega\frac{\left[F\left(x+\frac{d}{\sqrt{k}}\upsilon\right)-F(x)\right]}{(1/k)}d\pi_x(\upsilon)\\&\to DF(x)\text{ uniformly in }(x,\omega)\text{ as }c\to\infty.\end{aligned}$$

On the other hand, by Assumption II,

$$|(P^cDF)(x,\omega)-DF(x)|=\left|(DF)\left(x+\frac{d}{\sqrt{k}}\omega\right)-DF(x)\right|\to 0$$

uniformly in (x,ω) as $c\to\infty$.

Hence, for $F\in C^3_{\mathcal{K}}(G)$,

$$\|A^cP^cF-P^cDF\|\to 0 \quad\text{as}\quad c\to\infty.$$

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*) We also consider F as a function of (x,ω) by putting $F(x,\omega)=F(x)$.

Now let take $C_0(G)$, $C_0(S)$, P^c, T_t^D, T_t^c and $C^3_{\mathcal{K}}(G)$ for X, X_n, P_n, $T(t)$, $T_n(t)$ and M in our lemma. Then we get by the lemma

$$\|T_t^c P^c F - P^c T_t^D F\| \to 0 \text{ as } c\to\infty \text{ for any } F \in C_0(G).$$

Therefore $\|T_t^c F - {}^c T_t^D F\| \to 0$ as $c\to\infty$ for $F\in C_0(G)$, since $\|T_t^c P^c F - T_t^c F\| \to 0$ and $\|P^c T_t^D F - T_t^D F\| \to 0$ as $c\to\infty$. Thus we complete the proof.

2. Example. 2.1 (cf. [6]). Let $G=R^n$ and $\Omega={}^nS^{-1}$ be the $(n-1)$-dimensional unit sphere in R^n and π_x be the uniform probability measure on S^{n-1} (it is independent of $x\in R^n$). The transport process for this case is called *the isotropic scattering transport process with speed c*. Put $D=\frac{1}{2}\Delta$ and $d^2=\frac{2}{n}$. Then Assumptions I, II, and III are satisfied. Thus the isotropic scattering transport process converges to the Brownian motion.

2.2. $G=R^1$, $\Omega=\{-1, 1\}$, $\pi_x(\{1\})=\frac{1}{2}\left(1+\frac{1}{\sqrt{k}}\right)$, $\pi_x(\{-1\})=\frac{1}{2}\left(1-\frac{1}{\sqrt{k}}\right)$. $D=\frac{1}{2}\Delta+\frac{\partial}{\partial x}$, $d=1$.

2.3. $G=R^1$, $\Omega_x=\left\{-\frac{1}{a(x)}, \frac{1}{a(x)}\right\}(a(x)>0)$, $\pi_x\left(\left\{-\frac{1}{a(x)}\right\}\right)=\pi_x\left(\left\{\frac{1}{a(x)}\right\}\right)=\frac{1}{2}$. $D=\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}$, $d=1$.

References

[1] Ikeda, N., and H. Nomoto: Branching transport processes. Seminar on Prob., **25**, 63-104 (1966) (in Japanese).

[2] Monin, A. S.: A statistical interpretation of the scattering of macroscopic particles. Th. Prob. Appl., **3**, 298-311 (1956) (English transl.).

[3] Pinsky, M.: Differential equations with a small parameter and the central limit theorem for functions defined on a finite Markov chain. Z. Wahr. Geb., **9**, 101-111 (1968).

[4] Trotter, H. F.: Approximation of semigroups of operators. Pacific J. Math., **8**, 887-919 (1958).

[5] Watanabe, S., and To. Watanabe: Weak convergence of isotropic scattering transport process to Brownian motion (to appear).

[6] Watanabe, To.: Weak convergence of the isotropic scattering transport process with one speed in the plane to Brownian motion. Proc. Japan Acad., **44**, 677-680 (1968).