124. *On the Critical Points of Harmonic Functions*

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1. We admit as 2-cells the homeomorph of any convex polygon,[1] regarding the vertices and edges of this image as 0 and 1-cells respectively. An i-complex is a connected set of a finite number of i-cells ($i=1, 2$) and the characteristic ρ of a complex is defined as $\rho=-a_0+a_1-a_2$ where a_i is the number of i-cells ($i=0, 1, 2$) in the complex. The object of this paper is to give another proof to a theorem of Nevanlinna[2] on harmonic functions and to show that the characteristic of a domain plays an important rolle.

Let D be a domain or the union of a finite number of domains and $\bar{D}$ be its closure. We divide $\bar{D}$ in a finite number of 2-cells and consider $\bar{D}$ as a union of 2-complexes. We denote by a_i and a_i' the number of i-cells ($i=0, 1, 2$) contained in $\bar{D}$ and D respectively. Then $\rho(\bar{D})=-a_0+a_1-a_2$ and $\rho(D)=-a_0'+a_1'-a_2'$ are the sums of the characteristics of all connected components of $\bar{D}$ and D respectively. A 1-complex representing a simple closed curve has the same number of 0-cells as 1-cells and so contributes nothing to the characteristic. Hence we have $\rho(D)=\rho(\bar{D})$, when the boundary of D consists of a finite number of simple closed curves.

Let $u(z)$ be a harmonic function in a domain D and $C(u)$ be the niveau curve: $u(z)=\text{const.}=u$. The critical points of $u(z)$ in the ordinary sense are the points $z=x+iy$ at which $\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial y}=0$. Let $v(z)$ be the conjugate harmonic function of $u(z)$ and $w(z)=u(z)+iv(z)$. Then, by virtue of Cauchy-Riemann differential equation, such a point is a zero of $w'(z)$. The order of the zero is said to be the multiplicity of a critical point. Let $k-1$ be the multiplicity of a critical point z_0 of $u(z)$, then the niveau curve $C(u)$ through z_0 consists of k curves neighbouring z_0, each making an angle of $\dfrac{\pi}{k}$ at z_0 with its successor.

2. Let D be a domain bounded by m simple closed curves $C_1, C_2, \cdots, C_m$ and α be a set of a finite number of arcs on the boundary of D. We denote by n the number of arcs contained in α, which do not coinside with any of the whole curve C_i. Let $u(z)=\omega(z, \alpha, D)$ be the harmonic measure of α at the point z in D. We have $0<u(z)<1$ in D

and the boundary values of $u(z)$ are 1 on α and 0 outside α.

Theorem. *The sum of the multiplicities of the critical points of the harmonic measure $u(z)$ in the domain D is equal to $m+n-2$.*

Proof. Let $D(u)$ be the set of points z such that $u<u(z)<1$ and $\bar{D}(u)$ be its closure. The set $D(u)$ consists of n simply connected domains and some doubly connected domains for u which is sufficiently near to 1. These simply connected domains are bounded by the niveau curve $C(u)$ and an arc contained in α. The doubly connected domains are bounded by a simple closed component of $C(u)$ and a component of α which coinsides with some C_i. The characteristic of a simply connected domain is -1 and that of doubly connected domain is zero. Hence we have $\rho(D(u_0))=-n$ for u_0 which is sufficiently near to 1.

When there is no critical point on the niveau curve $C(u)$, the boundary of $D(u)$ consists of a finite number of simple closed curve and so we have $\rho(D(u))=\rho(\bar{D}(u))$.

When there is a critical point of multiplicity $k-1$ on the niveau curve $C(u)$. The critical point appears in the 1-complex representing $C(u)$ k times as 0-cell. Hence we have $\rho(\bar{D}(u))=\rho(D(u))+k-1$.

When there is no critical point in the open set $D(u_2)-\bar{D}(u_1)$ $(u_2<u_1)$, any component of the niveau curve $C(u)$ $(u_2<u<u_1)$ is a simple closed curve or a simple arc which combines two end points of α. Hence the set $D(u_2)-\bar{D}(u_1)$ consists of a finite number of simply or doubly connected domains and the number of simply connected components contained in $D(u_2)-\bar{D}(u_1)$ increased by the number of 1-cells contained in the union of 1-complexes representing α is equal to the number of 0-cells contained in the union of 1-complexes representing α. Hence the complex representing α contributes the same number to the characteristic as the complex representing $D(u_2)-\bar{D}(u_1)$. Since $D(u_2)+\alpha=\bar{D}(u_1)+(D(u_2)-\bar{D}(u_1))$, we have $\rho(\bar{D}(u_1))=\rho(D(u_2))$.

Therefore the only changes in $\rho(D(u))$ are caused by changing from $\rho(D(u))$ to $\rho(\bar{D}(u))$ when u is a level of a critical point. The set $D(0)$ is the whole domain D whose characteristic is $m-2$. Hence the sum of the multiplicities of the critical points of $u(z)$ in D is equal to $\rho(D(0))-\rho(D(u_0))=m+n-2$.

References

[1] M. Morse: Topological Methods in the Theory of Functions of a Complex Variable. Princeton (1947).
[2] R. Nevanlinna: Eindeutige analytische Funktionen. Berlin (1936).