187. Two Perturbation Theorems for Contraction Semigroups in a Hilbert Space

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The purpose of this note is to prove two perturbation theorems for contraction semigroups in a Hilbert space. A linear operator T in a Hilbert space H is said to be *accretive* if $\mathrm{Re}\,(Tu, u) \geqq 0$ for all $u \in \mathrm{D}(T)$ ($\mathrm{D}(T)$ denotes the domain of T). If T satisfies the conditions that $(T+\xi)^{-1} \in \mathscr{B}(\mathrm{H})$ ($\mathscr{B}(\mathrm{H})$ is the set of all bounded operators from H to H) and $\|(T+\xi)^{-1}\| \leqq \xi^{-1}$ for $\xi > 0$, T is said to be *m-accretive*. Let T and A be operators in H such that

$$(1) \qquad \|Au\| \leqq a\|u\| + b\|Tu\|, \; u \in \mathrm{D}(T) \subset \mathrm{D}(A),$$

where a, b are nonnegative constants. Then we say that A is *relatively bounded with respect to* T or simply *T-bounded*. The condition (1) is equivalent to

$$(2) \qquad \|Au\|^2 = a'^2\|u\|^2 + b'^2\|Tu\|^2, \; u \in \mathrm{D}(T) \subset \mathrm{D}(A),$$

and (2) is more convenient to our purposes. Now let T be m-accretive and A be accretive. Then it is known that $T+A$ is also m-accretive if A is T-bounded, with $b<1$ (Cf. E. Nelson [4] and K. Gustafson [1] for Banach space case. See also T. Kato [2], p. 499 and I. Miyadera [3]). Our first result is concerned with the case that $b'=1$ in a Hilbert space (but which does not cover the case that $a \neq 0$, $b=1$). The second result is concerned with a kind of large perturbation. Since $-T$ generates a contraction semigroup if and only if T is m-accretive, these results are considered as a part of the perturbation theory for contraction semigroups. And our results might have some applications in the theory of partial differential equations as shown by the example below.

For the later use we recall here some properties of m-accretive operators in H. First we note that an m-accretive operator is accretive and densely defined (cf. [2], p. 279).

Lemma. *Let T be densely defined and accretive. In order that T have the m-accretive closure, it is necessary that the range $\mathrm{R}(T+\xi)$ of $T+\xi$ be dense in* H *for every $\xi>0$, and it is sufficient that this be true for some $\xi>0$.*

Proof. The necessity is clear by the definition of the m-accretive operator, so that we prove the sufficiency part. Since T is densely defined and accretive, T is closable (see [2], p. 268, Theorem 3.4).

Since $\mathrm{Re}(Tu, u) \geqq 0$ for all $u \in \mathrm{D}(T)$, we have $\|(T+\xi)u\|^2 \geqq \xi^2 \|u\|^2$ for any $\xi > 0$. Therefore $T+\xi$ is invertible and the inverse $(T+\xi)^{-1}$ satisfies the inequality $\|(T+\xi)^{-1}\| \leqq \xi^{-1}$ for $\xi > 0$. But since the deficiency index of the closure of T is constant (see [2], p. 268, Theorem 3.2), $\mathrm{R}(T+\xi)$ is dense in H for any $\xi > 0$. Therefore the closure of T is m-accretive.

For other terminologies and notations appearing below see T. Kato [2].

1. The first result is given by

Theorem 1. *Let T and A be operators in H and T be m-accretive. If A is an accretive operator with $\mathrm{D}(A) \supset \mathrm{D}(T)$ and $A(T+\xi_0)^{-1}$ is a contraction for some $\xi_0 > 0$, then the closure S of $T+A$ is also m-accretive.*

Proof. Since $T+A$ is densely defined and accretive, by the Lemma above, it suffices to show that $\mathrm{R}(T+A+\xi_0)$ is dense in H. But since $T+A+\xi_0 = [1+A(T+\xi_0)^{-1}](T+\xi_0)$, it suffices to show that $\mathrm{R}(1-B)$ is dense in H, where $B = -A(T+\xi_0)^{-1} \in \mathscr{B}(\mathrm{H})$ and $\|B\| \leqq 1$.

To see this, it suffices to show that an element v of H orthogonal to this range must be zero. Now such a v satisfies $B^*v = v$. But since $B \in \mathscr{B}(\mathrm{H})$ and $\|B\| \leqq 1$, $B^*v = v$ is equivalent to $Bv = v$ (see [2], p. 290), that is, $A(T+\xi_0)^{-1}v + v = 0$. Setting $u = (T+\xi_0)^{-1}v \in \mathrm{D}(T)$, we have $(T+A+\xi_0)u = 0$. Since $T+A$ is accretive and $\xi_0 > 0$, this gives $u = 0$ and hence $v = 0$.

Corollary 1. *Let T be m-accretive and A be accretive. If A is T-bounded and* (2) *holds with $b' = 1$, that is,*

$$\|Au\|^2 \leqq a'^2 \|u\|^2 + \|Tu\|^2, \quad a' \geqq 0, \quad u \in \mathrm{D}(T) \subset \mathrm{D}(A), \tag{3}$$

then the closure S of $T+A$ is also m-accretive.

Remark. Corollary 1 is a slight generalization of the perturbation theorem for (essentially) selfadjoint operators by T. Kato (see [2], p. 289).

Proof of Corollary 1. It suffices to show that $A(T+\xi)^{-1}$ is a contraction for $\xi > a'$. Since T is accretive, (3) can be written

$$\begin{aligned} \|Au\|^2 &\leqq a'^2 \|u\|^2 + \|Tu\|^2 + (\xi^2 - a'^2)\|u\|^2 \\ &\leqq \|(T+\xi)u\|^2, \quad \xi > a'. \end{aligned}$$

Hence $A(T+\xi)^{-1} \in \mathscr{B}(\mathrm{H})$ and $\|A(T+\xi)^{-1}\| \leqq 1$ for $\xi > a'$. (If $a' \neq 0$, this is true for $\xi \geqq a'$.)

An accretive operator T in H is said to be *sectorial* with a vertex 0 and a semi-angle $\pi/2 - \omega$ if $e^{i\theta}T$ is also accretive for $-\omega \leqq \theta \leqq \omega$, $0 < \omega \leqq \pi/2$. Then the numerical range of T is a subset of a sector $|\arg \zeta| \leqq \pi/2 - \omega$. If $e^{i\theta}T$ is m-accretive for all $|\theta| \leqq \omega$, T is said to be *m-sectorial* with a vertex 0 and a semi-angle $\pi/2 - \omega$, and then $-T$ is the generator of a contraction holomorphic semigroup (cf. [2], p. 490). The following corollary is concerned with such operators.

Corollary 2. *Let T be m-sectorial with a vertex* 0 *and a semi-angle* $\pi/2-\omega$, *and let A be sectorial with a vertex* 0 *and a semi-angle* $\pi/2-\omega'$. *If the condition* (3) *is satisfied, then the closure S of $T+A$ is also m-sectorial with a vertex* 0 *and a semi-angle* $\pi/2-\omega''$, $\omega''=\min(\omega, \omega')$, *and* e^{-tS} *is holomorphic for* $|\arg t|<\omega''$ *and is a contraction.*

2. Our second result is the following

Theorem 2. *Let T and A be operators in* H *and T be m-accretive. If A is an accretive operator satisfying the condition*

(4) $$\mathrm{Re}\,(Tu, Au)\geqq 0 \quad \textit{for all}\ u\in \mathrm{D}(T)\subset \mathrm{D}(A),$$

then $T+A$ is m-accretive.

Proof. Since A is closable and $\mathrm{D}(A)\supset\mathrm{D}(T)$, there exists some constant $b>0$ such that $\|Au\|^2\leqq b^2(\|u\|^2+\|Tu\|^2)$ for all $u\in \mathrm{D}(T)$ (see [2], p. 191). Here, the case that $b\geqq 1$ is of some interests in the perturbation theory. It is known that $T+\varepsilon A$ is m-accretive if $b\varepsilon<1$.

Now adding $b^2k^2\varepsilon^4\|Au\|^2$ $(k=1, 2, \cdots)$ to the both sides of

$$\varepsilon^2\|Au\|^2\leqq b^2\varepsilon^2\|u\|^2+b^2\varepsilon^2\|Tu\|^2,$$

we have

$$\begin{aligned}\varepsilon^2(1+b^2k^2\varepsilon^2)\|Au\|^2 &\leqq b^2\varepsilon^2\|u\|^2+b^2\varepsilon^2(\|Tu\|^2+k^2\varepsilon^2\|Au\|^2)\\ &\leqq b^2\varepsilon^2\|u\|^2+b^2\varepsilon^2\|(T+k\varepsilon A)u\|^2.\end{aligned}$$

This gives the basic inequality in our argument

(5) $$\|\varepsilon Au\|\leqq b\varepsilon(1+b^2k^2\varepsilon^2)^{-1/2}(\|u\|+\|(T+k\varepsilon A)u\|),\quad k=0, 1, 2, \cdots.$$

Since we have known that $T+\varepsilon A$ is m-accretive, it follows from (5) for $k=1$ that $T+2\varepsilon A$ is m-accretive. Continuing this process for $k=2, 3, \cdots$, we can show that $T+k\varepsilon A$ is m-accretive for any positive integer k. Since ε is arbitrary positive number with $b\varepsilon<1$, it follows that $T+A$ is m-accretive.

As a particular case of Theorem 2 we obtain the following

Corollary. *Let T be selfadjoint and A be symmetric. If A satisfies the condition* (4), *then $T+A$ is selfadjoint.*

Proof. Since both iT and $-iT$ are m-accretive, and both iA and $-iA$ are accretive, it follows from Theorem 2 that $i(T+A)$ and $-i(T+A)$ are m-accretive. Therefore $T+A$ is selfadjoint.

Remark. If the assumptions of Theorem 2 are satisfied, we have $\mathrm{Re}\,((T+\xi)u, Au)\geqq 0$ for all $u\in\mathrm{D}(T)$. Therefore $A(T+\xi)^{-1}\in\mathscr{B}(\mathrm{H})$ is accretive for any $\xi>0$.

The method in the proof of Theorem 2 is applicable to the relatively bounded perturbation of some other kinds of operators satisfying the condition (4) in a Hilbert space. In fact, by the assumption (4), we can put off the restriction on a and b (see e.g. [2], p. 190, p. 196 and p. 497).

Example. Let $\mathrm{H}=L^2(-\infty, \infty)$, $T=-d/dx$ and $A=-f(x)d/dx$

where $f(x)=0$ for $x<0, f(x)=b>0$ for $x\geqq 0$. Then we have $2\mathrm{Re}(Au, u)=b|u(0)|^2\geqq 0$, $\mathrm{D}(A)\supset \mathrm{D}(T)$ and $(Tu, Au)=b\int_0^\infty |u'(x)|^2dx\geqq 0$. Hence it follows from Theorem 2 that $T+A$ is m-accretive for any $b>0$. Thus we see that the Cauchy problem $du/dt+(T+A)u=0$, $u(0)=u_0\in \mathrm{D}(T)$, is well-posed. This result is also easily verified by the method of characteristics in the theory of partial differential equations.

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