183. *Elliptic Modular Surfaces. II*

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In the first part [6], we have introduced a special class of elliptic surfaces called elliptic modular surfaces. In this part II, we shall indicate the proof of the theorem announced in [6] (Theorems 3.1 and 5.4). A reformulation and a few remarks will be given in Section 6.

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Notation. We use the same notations as in [6]. In particular, Γ always denotes a torsion-free subgroup of finite index of $SL(2, \boldsymbol{Z})$ (except in Remark 6.6).

3. The group of sections. In this section we shall prove

Theorem 3.1. *An elliptic modular surface has only a finite number of sections over the base curve.*

We denote by μ the index of the subgroup $\Gamma\{\pm 1_2\}$ in $SL(2, \boldsymbol{Z})$, and by t_1 (or t_2) the number of cusps of the first (or second) kind; put $t=t_1+t_2$. Then the genus g of the curve $\varDelta=\varDelta_\Gamma$ is given by the formula: $2g-2+t=\mu/6$. The index μ is clearly equal to the order of the meromorphic function J on $\varDelta$, the functional invariant of the elliptic modular surface B_Γ. Hence, from Theorem 12.2 of [1], we can compute the arithmetic and geometric genus of B_Γ.

Lemma 3.2. $p_a=\mu/12+t_2/2-1,$
$p_g=2g-2+t-t_1/2.$

Comparing Lemma 3.2 with Theorem 1.2 and Corollary 1.4, we get

Lemma 3.3. $r=0$ *and* $r'=2p_g$.

Thus the group $H^0(\varDelta, \Omega(B_0^\sharp))$ is of rank 0, i.e., finite. By considering the exact sequence ([1], Section 11)

$$(***) \qquad 0\to\Omega(B_0^\sharp)\to\Omega(B^\sharp)\to Q\to 0,$$

where the quotient Q is a sheaf of finite groups with the support on the finite set $\varDelta-\varDelta'$, we conclude that the group $H^0(\varDelta, \Omega(B^\sharp))$ is also finite, which completes the proof of Theorem 3.1.

Example 3.4. For the elliptic modular surface $B(N)$ for level N $(N\geq 3)$ (cf. Example 2.1—where we used the above Lemma 3.3), we can show that the group of sections of $B(N)$ is isomorphic to the finite group $(\boldsymbol{Z}/N\boldsymbol{Z})^2$. Moreover any two distinct sections do not meet each other. When $N=3$, $B(3)$ is a rational surface and the 9 sections are mutually disjoint exceptional curves of the first kind (cf. [6a]. p. 464).

For $N \geq 4$, the surfaces $B(N)$ are non-rational and minimal; $B(4)$ is a $K3$ surface mentioned at the end of Introduction of [6].

Remark. 3.5. The number r' is, by definition, the $\boldsymbol{Z}$-rank of the image $i^*H^1(\Delta, G)$ in $H^1(\Delta, \mathcal{O}(\mathfrak{f}))$, the latter being a complex vector space of dimension p_g ([1], p. 15). In view of the relation $r' = 2p_g$ of Lemma 3.3, it is natural to ask whether or not $i^*H^1(\Delta, G)$ is a lattice in $H^1(\Delta, \mathcal{O}(\mathfrak{f}))$. With this question in mind, we observed that the geometric genus p_g is equal to the dimension of the space of Γ-cusp forms of weight 3, which suggested the possible connection of our problem with Shimura's theory [3]. Cf. Section 5.

4. Parabolic cohomology and cusp forms. Let Γ be, as before, a torsion-free subgroup of finite index of $SL(2, \boldsymbol{Z})$. (In particular Γ does not contain the element -1_2.) Following Shimura [3], we define parabolic cohomology groups of Γ as follows. The group Γ acting on $\boldsymbol{Z}^2$ (from the left), an *integral parabolic cocycle* of Γ is a map $\mathfrak{x}$ of Γ into $\boldsymbol{Z}^2$ satisfying the 2 conditions:

i) $\mathfrak{x}(\sigma\sigma') = \mathfrak{x}(\sigma) + \sigma\mathfrak{x}(\sigma')$ for $\sigma, \sigma' \in \Gamma$;

ii) $\mathfrak{x}(\gamma) \in (\gamma - 1_2)\boldsymbol{Z}^2$ for parabolic γ of Γ.

A coboundary is a cocycle $\mathfrak{x}$ of the form: $\mathfrak{x}(\sigma) = (\sigma - 1_2)\mathfrak{x}_0$ for all σ in Γ with a fixed $\mathfrak{x}_0$ in $\boldsymbol{Z}^2$. The corresponding cohomology group is denoted by $H^1_{\mathrm{Par}}(\Gamma, \boldsymbol{Z}^2)$. Replacing $\boldsymbol{Z}$ by the real numbers $\boldsymbol{R}$ in the above, we get $H^1_{\mathrm{Par}}(\Gamma, \boldsymbol{R}^2)$ and a canonical homomorphism c of $H^1_{\mathrm{Par}}(\Gamma, \boldsymbol{Z}^2)$ into $H^1_{\mathrm{Par}}(\Gamma, \boldsymbol{R}^2)$. The following is a special case of Proposition 1, § 3 of [3].

Lemma 4.1. *The image of $H^1_{\mathrm{Par}}(\Gamma, \boldsymbol{Z}^2)$ is a lattice in the real vector space $H^1_{\mathrm{Par}}(\Gamma, \boldsymbol{R}^2)$.*

Next we denote by $\mathfrak{S}_m(\Gamma)$ the space of Γ-cusp forms of weight m. There is a hermitian metric on $\mathfrak{S}_m(\Gamma)$ called the Petersson metric: for f, g in $\mathfrak{S}_m(\Gamma)$,

$$(f, g) = \int_{\Gamma \backslash \mathfrak{H}} f(z)\overline{g(z)}y^{m-2}dxdy, \qquad z = x + iy \in \mathfrak{H}.$$

We are interested in the case where $m = 3$. For each f in $\mathfrak{S}_m(\Gamma)$, consider the "period" of the vector-valued differential form $\begin{pmatrix} z \\ 1 \end{pmatrix} f(z)dz$ on $\mathfrak{H}$:

$$\mathfrak{x}_f(\sigma) = \int_{z_0}^{\sigma \cdot z_0} \begin{pmatrix} z \\ 1 \end{pmatrix} f(z)dz, \qquad \sigma \in \Gamma,$$

where z_0 is a base point in $\mathfrak{H}$. $\mathfrak{x}_f$ is a $\boldsymbol{C}$-valued parabolic cocycle of Γ. We define $\varphi(f)$ as the cohomology class in $H^1_{\mathrm{Par}}(\Gamma, \boldsymbol{R}^2)$ containing the real cocycle $\mathrm{Re}(\mathfrak{x}_f)$. By a slight modification of the argument in § 5, [3], one can prove

Lemma 4.2. *φ is an isomorphism of $\mathfrak{S}_3(\Gamma)$ onto $H^1_{\mathrm{Par}}(\Gamma, \boldsymbol{R}^2)$.*

Remark 4.3. Combining Lemmas 4.1 and 4.2, one gets a complex torus attached to cusp forms of weight 3. In even weight case, Shimura [3] shows that these complex tori have structure of abelian varieties, by considering the imaginary part of the Petersson metric. This method cannot be applied to our case. But according to Shimura, it can be shown that our complex tori also have structure of abelian varieties for certain Γ, by considering their endomorphism algebras.

5. **Main results.** Let B be the elliptic modular surface attached to Γ. The line bundle $\mathfrak{f}$, defined in Section 1 [6], has the following description for our B. The group Γ acts on the product $\mathfrak{H}\times \boldsymbol{C}$ of the upper half plane and the complex plane by

$$\gamma : (z, \zeta)\mapsto(\gamma\cdot z, (cz+d)^{-1}\zeta), \qquad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Then the restriction of $\mathfrak{f}$ to the open set $\Delta'=\Gamma\backslash\mathfrak{H}$ is isomorphic to the quotient $\Gamma\backslash(\mathfrak{H}\times \boldsymbol{C})$ of $\mathfrak{H}\times \boldsymbol{C}$ by Γ (cf. [1], Section 11). Moreover, as is well known, the sheaf $\mathcal{O}(\mathfrak{k})$ of germs of sections of the canonical bundle $\mathfrak{k}$ on Δ is isomorphic to the sheaf of cusp forms of weight 2. This suggests

Lemma 5.1. *There is a canonical isomorphism (over $\boldsymbol{C}$):*

$$H^0(\Delta, \mathcal{O}(\mathfrak{k}-\mathfrak{f}))\simeq\mathfrak{S}_3(\Gamma).$$

We identify the two spaces by the canonical isomorphism.

By the duality theorem on a curve, there is a $\boldsymbol{C}$-bilinear non-degenerate pairing:

$$H^0(\Delta, \mathcal{O}(\mathfrak{k}-\mathfrak{f}))\times H^1(\Delta, \mathcal{O}(\mathfrak{f}))\to \boldsymbol{C}, \qquad (f, \xi)\mapsto\langle f, \xi\rangle.$$

On the other hand, the space $\mathfrak{S}_3(\Gamma)$ of cusp forms of weight 3 is self-dual (over $\boldsymbol{R}$) with respect to the Petersson metric. Hence

Lemma 5.2. *There is a sesqui-linear isomorphism*

$$\psi : H^1(\Delta, \mathcal{O}(\mathfrak{f}))\xrightarrow{\sim}\mathfrak{S}_3(\Gamma),$$

such that $\langle f, \xi\rangle=4(f, \psi(\xi))$ for all f in $\mathfrak{S}_3(\Gamma)$.

Combining the above with the results in the preceding sections, we get the following diagram:

$$\begin{array}{ccc} H^1(\Delta, G) & \xrightarrow{i^*} & H^1(\Delta, \mathcal{O}(\mathfrak{f})) & \\ \Big\downarrow{\scriptstyle\eta} & & \wr\Big\downarrow{\scriptstyle\psi} & \text{(Lemma 5.2)} \\ & & \mathfrak{S}_3(\Gamma) & \\ & & \wr\Big\downarrow{\scriptstyle\varphi} & \text{(Lemma 4.2)} \\ H^1_{\mathrm{Par}}(\Gamma, \boldsymbol{Z}^2) & \xrightarrow{c} & H^1_{\mathrm{Par}}(\Gamma, \boldsymbol{R}^2) & \end{array}$$

(image lattice; Lemma 4.1)

Theorem 5.3. *There is an isomorphism η of $H^1(\Delta, G)$ onto $H^1_{\mathrm{Par}}(\Gamma, \boldsymbol{Z}^2)$, which makes the above diagram commute.*

We have explicitly constructed the isomorphism η in terms of standard generators of the Fuchsian group Γ. It would presumably be the canonical isomorphism coming from the spectral sequence connecting the cohomologies of a discontinuous group with the cohomologies of the quotient space. From the exact sequence (**) of Section 1, we obtain

Theorem 5.4. *For an elliptic modular surface B, the group $H^1(\Delta, \Omega(B_0^\sharp))$ is isomorphic to the product of the complex torus $H^1(\Delta, \mathcal{O}(\mathfrak{f}))/i^*H^1(\Delta, G)$ and the finite group $H^2(\Delta, G)$.*

As for the group $H^1(\Delta, \Omega(B^\sharp))$, we see from the exact sequence (***) of Section 3 that it is a quotient of $H^1(\Delta, \Omega(B_0^\sharp))$ by a finite group. Thus $H^1(\Delta, \Omega(B^\sharp))$ is also a product of a complex torus and a finite group. According to a general result of M. Artin quoted in [2], the torsion subgroup of $H^1(\Delta, \Omega(B^\sharp))$ is divisible (cf. Lemma 6.2 below). Hence

Theorem 5.5. *For an elliptic modular surface B, the group $H^1(\Delta, \Omega(B^\sharp))$ has a structure of a complex torus of dimension p_g.*

6. Consequences and remarks. Given an elliptic modular surface $B=B_\Gamma$, we denote by $\mathcal{F}(\Gamma)=\mathcal{F}(J, G)$ the family of elliptic surfaces over Δ_Γ having the same functional and homological invariants as B_Γ. In general, the family $\mathcal{F}(J, G)$ modulo $\Omega(B^\sharp)$-equivalence is parametrized by the cohomology group $H^1(\Delta, \Omega(B^\sharp))$ so that algebraic surfaces in that family correspond to the elements of finite order in $H^1(\Delta, \Omega(B^\sharp))$ (see [1] Theorems 10.1 and 11.5). As an immediate consequence of Theorem 5.5, we have the following result, which answers Kodaira's question (cf. Introduction) in our special case:

Corollary 6.1. *Algebraic surfaces are dense in the family $\mathcal{F}(\Gamma)$ containing an elliptic modular surface B_Γ.*

In general, let K be the function field of the algebraic curve Δ. Given an elliptic surface B over Δ with a global section, its generic fibre E is an elliptic curve defined over K with a K-rational point. We denote by Ш(Δ, E) the Tate-Šafarevič group; it is the group of locally trivial principal homogeneous spaces for E over K (cf. [5], § 3).

Lemma 6.2. *The group Ш(Δ, E) is isomorphic to the torsion subgroup of $H^1(\Delta, \Omega(B^\sharp))$.*

In the case where $B=B_\Gamma$ is the elliptic modular surface attached to Γ, $K=K_\Gamma$ is the field of Γ-automorphic functions. We call the generic fibre E_Γ of B_Γ over Δ_Γ *the* elliptic curve over the field K_Γ. Then we can restate our main results in the following way.

Theorem 6.3. Let E_Γ be the elliptic curve over the field K_Γ of Γ-automorphic functions. Then

(i) E_Γ has only a finite number of K_Γ-rational points.

(ii) The group Ш$(\Delta_\Gamma, E_\Gamma)$ is (isomorphic to) the group of division points of a p_g-dimensional complex torus T_Γ. In particular, Ш$(\Delta_\Gamma, E_\Gamma) \simeq (\boldsymbol{Q}/\boldsymbol{Z})^{2p_g}$.

Remark 6.4. If we replace the word "complex torus" by "abelian variety" in (ii) (cf. Remark 4.3), Theorem 6.3 is of algebraic nature. It will be not too hard to show the validity of the statement (i) (with a proper modification) in positive characteristic case. We wonder if the statement (ii) also holds in such a case.

Remark 6.5. One might expect that, in analogy to Theorem 6.3, the division points of Shimura's abelian variety attached to Γ-cusp forms of even weight m should be related to the algebraic principal homogeneous spaces for the abelian variety $E_\Gamma^{(m-2)}$, the $(m-2)$-fold product of E_Γ with itself, over K_Γ. We can see however that this is not the case. In fact, the group of locally trivial algebraic principal homogeneous spaces for $E_\Gamma^{(m-2)}$ is isomorphic to the group of division points of the product $T_\Gamma^{(m-2)}$ of our complex torus T_Γ in Theorem 6.3. It would be interesting to seek some interpretation of Simura's abelian varieties from our viewpoint (cf. [4], p. 292).

Remark 6.6. The idea in this paper can be applied to some other cases. For instance, let Γ be a Fuchsian group obtained from an indefinite division quaternion algebra over $\boldsymbol{Q}$; assume that Γ is torsion-free. Then it is known that $\Delta=\Gamma\backslash\mathfrak{H}$ is a compact Riemann surface and that there is an abelian scheme B over Δ of fibre dimension 2. By defining a vector bundle $\mathfrak{f}$ (of rank 2) and a sheaf G over Δ in a similar way, we can prove analogues of Theorems 3.1 and 5.4. Note that, in this case, $B=B^{\sharp}=B_0^{\sharp}$. If we *assume* Lemma 6.2, i.e., $Ш(\Delta, A) \simeq H^1(\Delta, \Omega(B))_{\text{tor}}$, A being the generic fibre of B over Δ, then it would follow that the group $Ш(\Delta, A)$ is *not* divisible in general for an abelian variety of dimension ≥ 2 over a function field.

References

[1] K. Kodaira: On compact analytic surfaces. II–III. Ann. of Math., **77**, 563–626; **78**, 1–40 (1963).

[2] A. P. Ogg: Cohomology of abelian varieties over function fields. Ann. of Math., **76**, 185–212 (1962).

[3] G. Shimura: Sur les intégrales attachées aux formes automorphes. Jour. Math. Soc. Japan, **11**, 291–311 (1959).

[4] ——: On Dirichlet series and abelian varieties attached to automorphic forms. Ann. of Math., **72**, 237–294 (1962).

[5] J. Tate: On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. Seminaire Bourbaki, no. **306**, 1–26 (1965–66).

[6a] J. Igusa: Fibre systems of Jacobian varieties. III. Fibre systems of elliptic curves. Amer. J. Math., **81**, 453–476 (1959).

[6] T. Shioda: Elliptic modular surfaces. I. Proc. Japan Acad., **45**, 786–790 (1969).