34. *On Locally Compact Abelian Groups with Dense Orbits under Continuous Affine Transformations*

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1. Introduction. Let G be a locally compact abelian group and let T be a continuous automorphism of G. Then the continuous affine transformation $T(a)$, where a is an element in G, is defined by $T(a)(x) = a\cdot T(x)$ for x in G. In this paper we shall study some topological properties of G which has a continuous affine transformation $T(a)$ such that there is an element w in G such that $\{T(a)^n(w) \mid n=0, \pm 1, \pm 2, \cdots\}$ is dense in G. More precisely, the study has been derived from the following problem. Can a continuous affine transformation of a locally compact but non-compact abelian group have a dense orbit? In the sequel the problem shall be solved negatively in a sense. Studies which are closely related to this problem appear in [2], [3], [4], [5] and [6].

2. Locally compact abelian groups with dense orbits.

Lemma. *Let T be a linear transformation of the n-dimensional real euclidean space R^n onto itself. Then any affine transformation $T(a)$ $(a \in R^n)$ has no dense orbit in R^n except for the trivial case $n=0$.*

Proof. T can be considered as the linear tansformation of the n-dimensional complex euclidean space K^n onto itself in the natural way. Then from the matrix theory T can be represented by a triangular matrix under some suitable basis $\{e_1, e_2, \cdots, e_n\}$ of K^n.

$$T=\begin{pmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

An elementary calculation shows that T^{-1} is also represented by the following triangular matrix under the same basis $\{e_1, e_2, \cdots, e_n\}$.

$$T^{-1}=\begin{pmatrix} \lambda_1^{-1} & & & * \\ & \lambda_2^{-1} & & \\ & & \ddots & \\ 0 & & & \lambda_n^{-1} \end{pmatrix}$$

Fix elements a and w in R^n and let

$$a=\alpha_1 e_1+\alpha_2 e_2+\cdots+\alpha_p e_p, \qquad \alpha_i \in K \text{ for } i=1,2\cdots,p$$

and

$$w=\beta_1 e_1+\beta_2 e_2+\cdots+\beta_q e_q, \qquad \beta_j \in K \text{ for } j=1,2,\cdots,q,$$

where $\alpha_p \neq 0$ and $\beta_q \neq 0$. Then

$$T(a)(w) = (a_1 e_1 + \cdots + \alpha_p e_p) + (*e_1 + \cdots + *e_{q-1} + \beta_q \lambda_q e_q)$$

and

$$T(a)^{-1}(w) = (\gamma_1 e_1 + \cdots + \gamma_p e_p) + (*e_1 + \cdots + *e_{q-1} + \beta_q \lambda_q^{-1} e),$$

where $\gamma_1 e_1 + \cdots + \gamma_p e_p = T^{-1}(-a)$, so $\gamma_p = -\alpha_p \lambda_p^{-1} \neq 0$.

Put $r = \max \{p, q\}$.

Case I. If $\lambda_r \neq 1$ then

$$T(a)^k(w) = \begin{cases} *e_1 + \cdots + *e_{r-1} + \left[\lambda_r^k \left(\beta_r - \dfrac{\alpha_r}{1-\lambda_r}\right) + \dfrac{\alpha_r}{1-\lambda_r}\right] e_r \\ \quad \text{if } k \neq 0, \\ *e_1 + \cdots + *e_{r-1} + \alpha_r e_r \text{ if } k = 0. \end{cases}$$

This implies that if $|\lambda_r| \neq 1$ then the closure of $\{T(a)^k(w) \mid k = 0, \pm 1, \pm 2 \cdots\}$ is countable, whence the orbit of w under $T(a)$ can not be dense in R^n provided $|\lambda_r| \neq 1$. Thus if $\{T(a)^k(w) \mid k = 0, \pm 1, \pm 2, \cdots\}$ is dense in R^n then $|\lambda_r| = 1$ and $\{T(a)^k(w) + iT(a)^j(w) \mid k, j = 0, \pm 1, \pm 2, \cdots\}$ is dense in K^n. But in this case

$$\begin{aligned} T(a)^k(w) + iT(a)^j(w) &= *e_1 + \cdots + *e_{r-1} + \left[\lambda_r^k \left(\beta_r - \frac{\alpha_r}{1-\lambda_r}\right)\right. \\ &\quad \left. + \frac{\alpha_r}{1-\lambda_r}\right] e_r + i\left[\lambda_r^i \left(\beta_r - \frac{\alpha_r}{1-\lambda_r}\right) + \frac{\alpha_r}{1-\lambda_r}\right] e_r \\ &= *e_1 + \cdots + *e_{r-1} + \delta_r e_r. \end{aligned}$$

Therefore $|\delta_r|$ is bounded, which is impossible.

Case II. If $\lambda_r = 1$ then

$$T(a)^k(w) = \begin{cases} *e_1 + \cdots + *e_{r-1} + (\beta_r + k\alpha_r) e_r \text{ if } k \neq 0 \\ *e_1 + \cdots + *e_{r-1} + \alpha_r e_r \text{ if } k = 0. \end{cases}$$

Thus in order to see that $\{T(a)^k \mid k = 0, \pm 1, \pm 2, \cdots\}$ can not be dense in R^n, it suffices to apply an analogous argument as in Case I.

The proof is complete.

Theorem 1. *Let G be a locally compact abelian group and let $T(a)$ be a continuous affine transformation of G such that there is an element w in G such that $\{T(a)^n(w) \mid n = 0, \pm 1, \pm 2, \cdots\}$ is dense in G. If the connected component G_0 of the identity in G is not an open subgroup of G then G is compact.*

Proof. Since T is bi-continuous by [5, Lemma 2], G_0 is invariant under T. Let $\emptyset$ be the canonical map from G onto G/G_0 and let $\tilde{x}$ be a general element of G/G_0 such that $\emptyset(x) = \tilde{x}$ ($x \in G$). If $\tilde{T}$ is defined by $\tilde{T}(\tilde{x}) = \emptyset(T(x))$ for $\tilde{x}$ in G/G_0 then $\tilde{T}$ is well-defined and a continuous automorphism of G/G_0. It is clear that $\tilde{T}(\tilde{a})$ is a continuous affine transformation of G/G_0 such that $\{\tilde{T}(\tilde{a})^n(\tilde{w}) \mid n = 0, \pm 1, \pm 2, \cdots\}$ is dense in G/G_0. Since G_0 is not an open subgroup of G, G/G_0 is a totally disconnected non-discrete abelian group, thus G/G_0 is compact by [5, Theorem 1]. Let V be a neighborhood of the identity in G such that the closure of V is compact, and let H be the subgroup of G which is generated by V. Then H is an open subgroup of G, whence $G_0 \subset H$. Since G/G_0 is compact, G/H is finite. Thus it follows that G is compactly generated. The well-known structure theorem for a locally compact, compactly generated abelian group (see for instance [1, Theorem 9.8]) implies that G is topologically isomorphic with $R^p \times Z^q \times F$ for some nonnegative integers p and q and some compact abelian group F, where Z is the additive group of integers. But in the present case $q=0$, i.e., G is topologically isomorphic with $R^p \times F$. For if $q \neq 0$ then $G/R^p \times F = Z^q$ is not finite, which is not impossible since $R^p \times F$ is an open subgroup of G. Clearly F is invariant under T. So T induces a continuous automorphism T^* of $G/F = R^p$ such that the affine transformation $T^*(a^*)$ has a dense orbit $\{T^*(a^*)^n(w^*) \mid n=0, \pm 1, \pm 2, \cdots\}$ in R^p where a^* and w^* are elements in R^p such that $a \in a^*$ and $w \in w^*$, respectly. On the other hand, since T^* is a continuous automorphism of R^p, it satisfies $T^*(\alpha x^*) = \alpha T^*(x^*)$ for $\alpha \in R$ and $x^* \in R^p$, i.e., T^* is a linear transformation of the p-dimensional real euclidean space R^p onto itself. So by Lemma, $p=0$, i.e., G is topologically isomorphic with a compact abelian group F. This completes the proof.

Theorem 2. *Let G be a connected locally compact abelian group and let $T(a)$ be a continuous affine transformation of G such that there is an element w in G such that $\{T(a)^n(w) \mid n=0, \pm 1, \pm 2, \cdots\}$ is dense in G. Then G is compact.*

Proof. Since G is connected, it is compactly generated, whence it is topologically isomorphic with $R^p \times F$ for some nonnegative integer p and some compact abelian group F. Then the same argument as in the proof of Theorem 1 can be applied in order to prove that G is compact. The proof is complete.

The hypothesis that G_0 is not an open subgroup of G is necessary in Theorem 1, provided G is not connected (see [5, Remark 1]). But if T itself has a dense orbit in G then it is not necessary, i.e., we have the following.

Theorem 3. *Let G be a locally compact abelian group and let T be a continuous automorphism of G such that there is an element w in G such that $\{T^n(w) \mid n=0, \pm 1, \pm 2, \cdots\}$ is dense in G. Then G is compact.*

Proof. Let G_0 be the connected component of the identity in G. Then T induces a continuous automorphism $\tilde{T}$ of G/G_0 which has a dense orbit in G/G_0. Thus by [5, Theorem 3], G/G_0 is compact. Then it is a routine matter to show that G is topologically isomorphic with $R^p \times F$ for some nonnegative integer p and some compact abelian group F. The proof is now obvious.

Theorem 4. *Let G be a locally compact abelian group with a countable open basis and let $T(a)$ be a continuous affine transformation of G which is ergodic with respect to a Haar measure on G. Then G is compact whenever one of the following three statements is true:*

1) *The connected component G_0 of the identity e in G is not an open subgroup of G.*

2) *G is connected.*

3) *$T(a)$ is an automorphism, i.e., $a=e$.*

Proof. By the ergodicity of $T(a)$ and the second countability of G, the orbit of x under $T(a)$ is dense in G for almost all x in G. Thus Theorems 1, 2 or 3 can be applied in order to prove that G is compact. The proof is complete.

Theorem 5. *Let G be a locally compact abelian group which has an element a such that $\{a^n \mid n=0, \pm 1, \pm 2, \cdots\}$ is dense in G. Then G is compact whenever one of the following two statements is true:*

1) *The connected component G_0 of the identity in G is not an open subgroup of G.*

2) *G is connected.*

The proof is obvious from the above.

Remark 1. In Theorem 3 the hypothesis that G is abelian is not necessary, i.e., if G is a locally compact (not necessarily abelian) group which has a continuous automorphism with a dense orbit then G is compact. In order to prove this it suffices to apply analogous arguments as in [2] and [4], by virtue of Theorem 3 and [5, Theorem 3]. We omit the details here.

Remark 2. Concerning Theorem 4 it seems worth to notice that if G is a locally compact group which has an ergodic continuous automorphism with respect to a Haar measure on G then G is compact [4].

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