54. *Properties of Ergodic Affine Transformations of Locally Compact Groups. II*

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This is a continuation of the preceding paper [5]. The followings shall be proved below: (1) If G is a locally compact non-discrete abelian group which has an ergodic affine transformation S with respect to a Haar measure on G then G is compact. (2) An affine transformation S of a locally compact abelian group G which has a dense orbit in G is ergodic with respect to a Haar measure on G.

Theorem 1. *If G is a locally compact non-discrete abelian group which has an ergodic affine transformation $S(x)=a+T(x)$ with respect to a Haar measure μ on G then G is compact.*

Proof. Let G_0 be the connected component of the identity 0 of G. Since T is *bi*-continuous by virtue of [5, Theorem 1], G_0 is invariant under T. Thus S induces an affine transformation S_1 of G/G_0 in the following way

$$S_1(x+G_0)=a+T(x)+G_0 \qquad \text{for} \qquad x+G_0 \in G/G_0.$$

It is clear that S_1 is ergodic with respect to a Haar measure μ_1 on G/G_0.

Case I. Let G_0 be not open in G. Then G/G_0 is a locally compact totally disconnected non-discrete abelian group which has an ergodic affine transformation with respect to a Haar measure on G/G_0. Hence G/G_0 is compact by [5, Theorem 3], from which it follows easily that G is compactly generated. Thus the well-known structure theorem for a locally compact, compactly generated abelian group (see [1, Theorem (9.8)]) implies that G is topologically isomorphic with $R^p \times Z^q \times F$ for some nonnegative integers p and q and some compact abelian group F, where R is the real line and Z is the additive group of integers. But in the present case $q=0$, i.e., G is topologically isomorphic with $R^p \times F$. For if $q \neq 0$ then $G/(R^p \times F)=Z^q$ is not finite, which is impossible since $R^p \times F$ is an open subgroup of G. It is clear that F is invariant under T. So the ergodic affine transformation $S(x)=a+T(x)$ of G induces an affine transformation $S_2(x+F)=a+T(x)+F$ of $G/F=R^p$ which is ergodic with respect to a Haar measure on $G/F=R^p$. By [3, Theorem 4], $G/F=R^p$ is compact, therefore $G=F$, i.e., G is compact.

Case II. Let G_0 be open in G. Then G/G_0 is a discrete abelian group which has an ergodic affine transformation S with respect to a Haar measure on G/G_0. Hence we observe

$$\{S_1^n(a+G_0) \mid n=0, \pm 1, \pm 2, \cdots\} = G/G_0.$$

If $S^n(a+G_0) \neq a+G_0$ for all positive integers n, then S can not be ergodic. In fact, since G is not discrete and G_0 is open, there exist two nonvoid open sets U_1 and U_2 in G_0 such that $U_1 \cap U_2 = \phi$. The set

$$E = \bigcup_{n=-\infty}^{\infty} S^n(U_1)$$

is a Borel set such that $S^{-1}(E)=E$, $\mu(E)>0$ and $\mu(G \cap E^c) \geqq \mu(U_2)>0$. Thus the ergodicity of S implies that there exists a positive integer k for which $S^k(a+G_0)=a+G_0$, whence it follows

$$\{S_1^n(a+G_0) \mid n=0, 1, \cdots, k-1\} = G/G_0.$$

This demonstrates that G is compactly generated. So the same argument as in Case I suffices in order to prove that G is compact.

The proof is complete.

Theorem 2. *Let G be a locally compact abelian group. Suppose $S(x)=a+T(x)$ is an affine transformation of G such that there exists an element w in G such that $\{S^n(w) \mid n=0, \pm 1, \pm 2, \cdots\}$ is dense in G. Then S is ergodic with respect to a Haar measure on G.*

Proof. If G is discrete the proof is trivial, and so we assume that G is not discrete. Let G_0 be the connected component of the identity 0 of G.

Case I. Let G_0 be not open in G. Then it follows from [3, Theorem 1] that G is compact. Hence S is ergodic by virtue of [4, Theorem 9].

Case II. Let G_0 be open in G. Since T is *bi*-continuous by [2, Lemma 2], G_0 is invariant under T. Thus S induces an affine transformation S of G/G_0 as follows

$$S_1(x+G_0) = a+T(x)+G_0 \qquad \text{for} \qquad x+G_0 \in G/G_0.$$

It is clear that S_1 has a dense orbit $\{S_1^n(w+G_0) \mid n=0, \pm 1, \pm 2, \cdots\}$ in G/G_0. Since G/G_0 is discrete, this implies

$$\{S_1^n(w+G_0) \mid n=0, \pm 1, \pm 2, \cdots\} = G/G_0.$$

Thus analogous arguments as in the proof of Theorem 1 imply that G is topologically isomorphic with $R^p \times F$ for some nonnegative integer p and some compact abelian group F. Clearly F is invariant under T, and so S induces an affine transformation S_2 of $G/F=R^p$ as follows

$$S_2(x+F) = a+T(x)+F \qquad \text{for} \qquad x+F \in G/F=R^p.$$

Since S_2 has a dense orbit $\{S_2^n(w+F) \mid n=0, \pm 1, \pm 2, \cdots\}$ in $G/F=R^p$, it follows from [3, Theorem 2] that $G/F=R^p$ is compact, whence $G=F$. So S is ergodic in this case too.

The proof is complete.

Unsolved problems. (1) Let G be a connected locally compact non-abelian group which has an ergodic affine transformation or a dense orbit under an affine transformation. Then is G compact? If the answer is affirmative it can be shown by using Theorem 3 and [2, Theorem 1] that an affine transformation of a locally compact, non-compact, non-discrete group can not be ergodic or have a dense orbit.

(2) Let $S(x)=a\cdot T(x)$ be an affine transformation of a compact non-abelian group G which has a dense orbit in G. Then is S ergodic?

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