87. *Subnormal Weighted Shifts and the Halmos-Bram Criterion*

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The question of when a weighted shift is subnormal was treated by J. G. Stampfli in a paper called "which weighted shifts are subnormal" [1]. In that paper, an explicit matricial construction is given for the minimal normal extension, assuming the shift to be subnormal. An examination of when this construction is possible leads to conditions for subnormality in terms of the weights. The construction is not easy and the conditions are not especially transparent. However, the conditions he obtains enable him to answer certain questions such as the following: how many initial weights (of a subnormal shift) can be prescribed arbitrarily?

The purpose of this note is to give a criterion for subnormality rather different from Stampfli's. The questions he answers seem much more accessible from our point of view and reduce to elementary problems concerning the moments of measures. Professor Halmos has informed us that the connection between subnormality and moment sequences (our corollary) was noted previously by C. E. Berger but apparently this observation is unpublished.

In addition, the criterion given below bears on a general question relating to the well-known general criterion for subnormality due to Halmos and Bram (see [2]). Let's agree to call an infinite matrix $(a_{ij})_0^\infty$ nonnegative if all the principal finite sub-matrices $(a_{ij})_0^n$ are non-negative definite. A sequence of vectors $\{f_i\}_0^\infty$ will be called sub-normal if the matrix $(\langle W^j f_i, W^i f_j\rangle)_0^\infty$ is non-negative (all this is relative to a fixed operator W). The Halmos-Bram theorem says W is subnormal if and only if each sequence is. The practical drawback of this criterion is evident. Consequently, it would seem of interest to find a non-trivial class of operators for which subnormality could be checked by examining only a small number of sequences. The weighted shifts are such a family.

We let H be a separable Hilbert space. The spectral radius of a bounded operator T is $r(T)$. We will always assume that the weights of a weighted shift are positive.

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2. The criterion.

Theorem. *Let W be a weighted shift, $We_i=w_ie_{i+1}$, $i=0,1,\cdots$ where the $\{e_i\}_0^\infty$ is an o.n. basis of H. Then W is subnormal if (and, trivially, only if) the sequences $\{e_i\}_0^\infty$ and $\{e_{i+1}\}_0^\infty$ are subnormal.*

Proof. For $i,j>0$, $W^je_i=w_iw_{i+1}\cdots w_{i+j-1}e_{i+j}$ and so $\langle W^je_i, W^ie_j\rangle =(w_iw_{i+1}\cdots w_{i+j-1})(w_jw_{j+1}\cdots w_{i+j-1})$. Setting $f_i=e_{i+1}$, $\langle W^jf_i, W^if_j\rangle =(w_{i+1}\cdots w_{i+j})(w_{j+1}\cdots w_{i+j})$. Write $p_i=\prod_0^{i-1} w_k$ for $i>0$, $p_0=1$. The formulas above can be rewritten:

$$(\langle W^je_i, W^ie_j\rangle)_0^\infty=\left(\frac{p_{i+j}^2}{p_ip_j}\right)_0^\infty \quad\text{and}\quad (\langle W^jf_i, W^if_j\rangle)_0^\infty=\left(\frac{p_{i+j+1}^2}{p_{i+1}p_{j+1}}\right)_0^\infty.$$

(The case where i or j vanishes is easily verified).

Now recall that the matrices

$$(p_ip_j)_0^\infty,\ (p_{i+1}p_{j+1})_0^\infty,\ (p_i^{-1}p_j^{-1})_0^\infty,\quad\text{and}\quad (p_{i+1}^{-1}p_{j+1}^{-1})_0^\infty$$

are non-negative. Since the Hadamard (termwise) product of non-negative matrices is again non-negative, the hypotheses of the theorem are equivalent to the fact that

$$(q_{i+j})_0^\infty \quad\text{and}\quad (q_{i+j+1})_0^\infty,\ q_i=p_i^2$$

are non-negative. But this is precisely the condition for the solvability of the Stieltjes moment problem with moments $\{q_i\}_0^\infty$ (see [3]). Thus there is a positive Borel measure ν on $[0,\infty]$ such that

$$q_i=\int_0^\infty t^i d\nu(t),\ i=0,1,\cdots.$$

Since $\lim\sup q_n^{1/n}\le r^2(W)=R^2$, it follows that ν is concentrated on the interval $[0,R^2]$. Change variables to write

$$q_i=\int_0^R r^{2i}d\mu(r)$$

On the disc $\Delta_R=\{z:\ |z|\le R\}$, define the measure σ by

$$d\sigma=d\mu\times\frac{d\theta}{2\pi}$$

where $d\theta$ is angular Lebesgue measure. This definition is slightly abusive since Δ_R is not a product space as indicated and μ may have mass at 0, but confusion is unlikely to arise. Note that

$$\int_{\Delta_R} z^n z^m d\sigma=\delta_{mn}q_n.$$

Let M be the operator on $L_2(d\sigma,\Delta_R)$ of multiplication by z. Let $L_2^+(d\sigma,\Delta_R)$ be the closed span of $\{1,z,z^2,\cdots\}$. Then the mapping $U: H\to L_2^+$ defined by $Ue_n=z^n/p_n$ is an isomorphism setting up a unitary equivalence between W and $M|L_2^+$. Hence W is subnormal. This concludes the proof.

Corollary. *Let W be normalized so that $r(W)=1$. Then W is subnormal if and only if $\{q_i\}$ is a Hausdorff moment sequence.*

3. Some consequences.

In this section, we assume throughout that W is hyponormal (i.e. $W^*W \geq WW^*$). This is the same as assuming the weights are non-decreasing. Also, this is equivalent to the assumption that the matrices $(\langle W^j e_i, W^i e_j \rangle)$ and $(\langle W^j e_{i+1}, W^i e_{j+1} \rangle)$ have non-negative 2×2 principal minors.

Stampfli's results include these:

(a) If $w_0 < w_1 < w_2$, then weights $w_n, n>2$ may be found so that the associated W is subnormal.

(b) There exist numbers $0 < w_0 < w_1 < w_2 < w_3$ such that any W with $w_i (i=0,1,2,3)$ as initial weights is not subnormal.

(c) If for some $i, i\geq 1$, $w_i = w_{i+1}$, then $w_j = w_{j+1}$ for all $j \geq 1$ providing W is subnormal.

Proof of (c). Since W is subnormal, we may write $q_j = \int_0^R t^j d\mu(t)$. Since $w_k^2 = q_{k+1}/q_k$, we have $q_{i+1}^2 = q_i q_{i+2}$. Thus equality holds in the Schwarz inequality $\left(\int_0^R t^{i+1} d\mu\right)^2 \leq \left(\int_0^R t^i d\mu\right)\left(\int_0^R t^{i+2} d\mu\right)$. Thus $t^i = ct^{i+2}$ a.e. (μ), and so μ is a point mass at, say ξ, plus possibly a point mass at 0. Thus for any $j \geq 1$, $q_{j+1}/q_j = q_{j+2}/q_{j+1}$.

Proof of (b). It suffices to pick w_0, w_1, w_2, w_3 such that

$$q_1 < q_2/q_1 < q_3/q_2 < q_4/q_3$$

but

$$\det \begin{pmatrix} 1 & q_1 & q_2 \\ q_1 & q_2 & q_3 \\ q_2 & q_3 & q_4 \end{pmatrix} < 0$$

Pick, for example, $q_1 = 1$, $q_2 = 2$, $q_3 = 5$, $q_4 = 12\frac{3}{4}$.

Proof of (a). We are assuming that $q_1 < q_2/q_1 < q_3/q_2$. On the space P_3 of polynomials of degree 3 or less, define the functional L by $L(a_0 t^3 + a_1 t^2 + a_2 t + a_3) = a_0 q_3 + a_1 q_2 + a_2 q_1 + a_3$. It's elementary that

$$L[(t+a)^2] \geq q_2 - q_1^2 > 0 \quad \text{and} \quad L[t(t+a)^2] \geq (q_1 q_3 - q_2^2)/q_1 > 0.$$

Using this fact, it is very easy to show that for each $p \in P_3$ satisfying $p(t) \geq 0$ for $t \geq 0$ and $p \not\equiv 0$, we have $L(p) > 0$. Suppose, for each interval $[0, n]$, $n = 1, 2, \cdots$, there is a $p_n \in P_3$ such that $p_n(x) \geq 0$ for $x \in [0, n]$, $p_n \not\equiv 0$, and $L(p_n) \leq 0$. We may also suppose that the largest modulus of any coefficient in p_n is exactly 1. Then an obvious compactness argument gives a $p \in P_3$, $p(x) \geq 0$ for $x \geq 0$, $p \not\equiv 0$ but $L(p) \leq 0$, a contradiction. Hence, on some finite interval L is positive on P_3. Extend L to be positive on $C(I)$, $I = [0, a]$ so that $L(f) = \int_0^a f(t) d\mu$. Define $q_n = \int_0^a t^n d\mu$, $n > 3$. Since $\lim q_n^{1/n} \leq a$ and q_{n+1}/q_n is increasing, we have $q_{n+1}/q_n = 0(1)$ so $w_n = 0(1)$ and we are done.

References

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[3] J. A. Shohat and J. D. Tamarkin: The Problem of Moments. A.M.S. (1943).