86. *Connection of Topological Manifolds*

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(Comm. by Kinjirô KUNUGI, M. J. A., April 13, 1970)

Introduction. The notion of connection for topological fibre bundles has been introduced by the author ([1], [2]). Then a connection of a topological manifold X should be defined to be a connection of the tangent microbundle ([10]) of X. The purpose of this note is to show the existence of connection for any paracompact topological manifold and state some related topics. The details will appear in the Journal of the Faculty of Science, Shinshu University, Vol. 5, 1970.

1. Connection of topological fibre bundles. Let $\xi=\{g_{UV}(x)\}$ be a topological G-bundle over a normal paracompact space X, where G is a topological group, $\{g_{UV}(x)\}$ is the transition function of ξ with covering system $\{U\}$. Then a connection $\theta=\{s_U(x_0, x_1)\}$ of ξ is a collection of the germ (at the diagonal of $U\times U$) of G-valued function $s_U(x_0, x_1)$ such that

$$S_U(x, x)=e, \textit{ the unit of } G,$$
$$g_{UV}(x_0)s_V(x_0, x_1)g_{VU}(x_1)=s_U(x_0, x_1).$$

We denote by $\mathcal{G}=\mathcal{G}_G$ the sheaf of germs of the germ (at the diagonal of $X\times X$) of G-valued function $\{t_U(x_0, x_1)\}$ such that

$$t_U(x, x)=e,$$
$$g_{UV}(x_1)t_V(x_0, x_1)g_{VU}(x_1)=t_U(x_0, x_1),$$

then we can define a cohomology class $o(\xi)$ of $H^1(X, \mathcal{G})$ such that ξ has a connection if and only if $o(\xi)$ vanishes in $H^1(X, \mathcal{G})$ ([3]).

In fact, if G is either of

(i) *There is a topological ring $R\supset G$ such that there is a neighbourhood $U(e)$ of e in R which is contained in G,*

(ii) *G is a locally compact, connected, locally connected solvable group,*

then a G-bundle ξ has a connection ([1], [3]).

If $\theta=\{s_U(x_0, x_1)$ is a connection of ξ, then

$$\delta\theta=\{s_U(x_1, x_2)s_U(x_0, x_2)^{-1}s_U(x_0, x_1)\}$$

is called the curvature of θ. We can prove that if the value of $\delta\theta$ is contained in H, a subgroup of G, then the connected component of the structure group of ξ is reduced to H ([1], [2]).

Note 1. If $G=\boldsymbol{C}^*$, the multiplicative group of complex numbers without 0, then the Alexander-Spanier class of $\delta\theta$ is the 1-st (complex)

Chern class of ξ ([3]).

Note 2. Regarding the total space of the principal bundle of ξ to be a G-space E, we can define a connection of ξ to be a germ (at the diagonal of $E\times E$) of a G-valued function s such that

$$s(x, x)=e,$$
$$s(x\alpha, y\beta)=\alpha^{-1}s(x, y)\beta,\ x, y\in E,\ \alpha, \beta\in G.$$

2. Bundles which have no connections. Let $\mathcal{E}$ be a normal paracompact space, then we have the following exact sequence.

$$0\longrightarrow \boldsymbol{Z}(\mathcal{E})\xrightarrow{\ i\ } \boldsymbol{C}(\mathcal{E})\xrightarrow{\ j\ } \boldsymbol{C}^*(\mathcal{E})\xrightarrow{\ \delta\ } H^1(\mathcal{E}, \boldsymbol{Z})\longrightarrow 0,$$

where $\boldsymbol{Z}(\mathcal{E})$, $\boldsymbol{C}(\mathcal{E})$ and $\boldsymbol{C}^*(\mathcal{E})$ are the groups of continuous integer valued functions, complex valued functions and $\boldsymbol{C}^*$-valued functions on $\mathcal{E}$, i is the inclusion, j is given by

$$j(f)=\exp 2\pi\sqrt{-1}f.$$

We set $j(\boldsymbol{C}(\mathcal{E}))=\boldsymbol{C}_0^*(\mathcal{E})$. Then if X is a normal paracompact space, we have

$$H^p(X, \boldsymbol{C}_0^*(\mathcal{E}))=H^{p+1}(X, \boldsymbol{Z}(\mathcal{E})),\quad p\geqq 1,$$

and the exact sequence

$$\cdots\longrightarrow H^1(X, \boldsymbol{C}_0^*(\mathcal{E}))\xrightarrow{\ \iota\ } H^1(X,\ \boldsymbol{C}^*(\mathcal{E}))\xrightarrow{\ \delta^*\ }$$
$$\longrightarrow H^1(X, H^1(\mathcal{E}, \boldsymbol{Z}))\xrightarrow{\ \delta\ } H^2(X, \boldsymbol{C}_0^*(\mathcal{E}))\longrightarrow\cdots.$$

Here ι is the map induced from the inclusion. Then we can prove

Theorem. *Regarding $\boldsymbol{C}^*(\mathcal{E})$ to be a topological group by the compact open topology, a $\boldsymbol{C}^*(\mathcal{E})$-bundle over X has connection if and only if its class in $H^1(X, \boldsymbol{C}^*(\mathcal{E}))$ (defined by its transition functions) belongs in ι-image.*

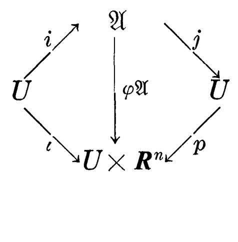
For example, if $X=\mathcal{E}=S^1$, then

$$H^1(S^1, \boldsymbol{C}_0^*(S^1))=H^2(S^1, \boldsymbol{Z})=0,$$
$$H^2(S^1, \boldsymbol{C}_0^*(S^1))=H^3(S^1, \boldsymbol{Z})=0,$$
$$H^1(S^1, H^1(S^1, \boldsymbol{Z}))=H^1(S^1, \boldsymbol{Z})=\boldsymbol{Z}.$$

Hence we get $H^1(S^1, C^*(S^1))=\boldsymbol{Z}$, and its non-trivial element does not belong in ι-image.

Therefore the equivalence classes of $C^*(S^1)$-bundles over S^1 are in 1–1 correspondence with $\boldsymbol{Z}$, and no non-trivial $C^*(S^1)$-bundle over S^1 has topological connections.

3. Connection of microbundles. A microbundle $\mathfrak{X}$ over X is a sequence $X\xrightarrow{\ i\ } E\xrightarrow{\ j\ }$ X with commutative diagram



$\iota(x)=x\times 0,\quad p(x, a)=x,$

where U and $\mathfrak{A}$ are the open sets of X and E ([10]). If X is a normal paracompact space, then a microbundle over X is regarded to be an $H_0(n)$-bundle over X ([6], [9]), where $H_0(n)$ is the group of all homeomorphisms of $\boldsymbol{R}^n$ which fix the origin with compact open topology. Similarly, a microbundle over a normal paracompact space is regarded to be an $H_*(n)$-bundle, where $H_*(n)$ is the group of the germs of the elements of $H_0(n)$ at the origin (for the detailed definition and the method of the construction of $H_*(n)$-bundles associated to a microbundle, see [4]).

The definition of connection of a microbundle $\mathfrak{X}$ should be different either we regard $\mathfrak{X}$ to be an $H_0(n)$-bundle or an $H_*(n)$-bundle, and the existence of a connection of $\mathfrak{X}$ as an $H_*(n)$-bundle follows from the existence of a connection of $\mathfrak{X}$ as an $H_0(n)$-bundle. But since we get

$$H^1(X, \mathcal{G}_{H_0(n)}) = H^1(X, \mathcal{G}_{H_*(n)}), \quad n \geqq 5,$$

under the natural map by virtue of the annulus theorem ([7], [12], see also [5]), we obtain

Lemma 1. *A microbundle $\mathfrak{X}$ over a normal paracompact space has a connection as an $H_0(n)$-bundle if and only if $\mathfrak{X}$ has a connection as an $H_*(n)$-bundle if $n \geqq 5$.*

Note. Since we know the homotopy types of $H_0(1)$ and $H_0(2)$ are $O(1)$ and $O(2)$. Lemma 1 is also true for $n=1, 2$.

4. Tangent microbundle. The tangent microbundle τ of a paracompact manifold X is given by the sequence

$$X \xrightarrow{\Delta} X \times X \xrightarrow{p} X, \quad \Delta(X) = (x, x), \quad p(x, y) = x,$$

with the local trivialization

$$\varphi_U(x, y) = (x, h_U(y) - h_U(x)),$$

where $\{U\}$ is an open covering of X, $\{h_U\}$, $h_U: U \to \boldsymbol{R}^n$, are the homeomorphisms by which the manifold structure of X is given ([10]). Then, setting

$$h_{U,x}(y) = h_U(y) - h_U(x),$$

the (representations of) transitions of τ as an $H_*(n)$-bundle ($n = \dim X$) are given by

$$g_{UV}(x) = h_{U,x} h_{V,y}^{-1}.$$

Then we get

$$h_{U,x_0}^{-1} s_U(x_0, x_1) h_{U,x_1} = h_{V,x_0}^{-1} s_V(x_0, x_1) h_{V,x_1},$$

if $\{s_U(x_0, x_1)\}$ is a connection of τ as an $H_*(n)$-bundle. Hence we obtain

Lemma 2. *The tangent microbundle τ of X has a connection as an $H_*(n)$-bundle if and only if there exists a homeomorphism $t(x_0, x_1)$ such that*

($*$) *$t(x, x)$ is the identity map on some neighbourhood of x,*

($**$) *$t(x_0, x_1)$ is a homeomorphism from a neighbourhood of x_1 to a neighbourhood of x_0 which maps x_1 to x_0,*

(***) *$t(x_0, x_1)$ depends continuously on x_0, x_1 (for the detailed definition of the continuity of t, cf.* [4]),
if x_0, x_1 belongs in $U(\Delta(X))$, where $U(\Delta(X))$ is a neighbourhood of $\Delta(X)$, the diagonal of $X\times X$, in $X\times X$.

5. Existence of $t(x_0, x_1)$. On U, a coordinate neighbourhood of X, we can construct $t(x_0, x_1)=t_U(x_0, x_1)$ by

$$(\sharp)\qquad t_U(x_0, x_1)(y) = h_U^{-1}(h_U(y)+h_U(x_0)-h_U(x_1)).$$

Then setting

$$r_{UV}(x)(y)=t_U(x, y)t_V(x, y)^{-1},$$

we can regard (the class of) $r_{UV}(x)$ is an element of $F_*(\boldsymbol{R}^n, H_*(n))$, the group of the germs at the origin of the continuous maps from $\boldsymbol{R}^n$ into $H_*(n)$ such that whose value at the origin is the identity of $H_*(n)$. Hence $\{r_{UV}(x)\}$ defines an $F_*(\boldsymbol{R}^n, H_*(n))$-bundle over X (the definition and the method of the construction of an $F_*(\boldsymbol{R}^n, H_*(n))$-bundle is similar that of an $H_*(n)$-bundle, cf. [4]). Then since $F_*(\boldsymbol{R}^n, H_*(n))$ is contractible and X is paracompact normal, there exists a collection of continuous maps $\{q_U(x)\}$, $q_U(x): U\rightarrow F_*(R^n, H_*(n))$, such that

$$r_{UV}(x)=q_U(x)q_V(x)^{-1},$$

([8], [11]). Hence we get

$$q_U(x, y)^{-1}t_U(x, y)=q_V(x, y)^{-1}t_V(x, y),$$

where $q_U(x, y)$ is given by $q_U(x)(y)=q_U(x)(h_{U,x}^{-1}(y)$. Therefore, setting

$$t(x_0, x_1)\,|\,U=q_U(x_0, x_1)^{-1}t_U(x_0, x_1),$$

we obtain the existence of $t(x_0, x_1)$ which satisfies the conditions of Lemma 2. Hence we have

Theorem. *If X is a normal paracompact (topological) manifold, then X has a topological connection if we regard τ, the tangent microbundle of X, to be an $H_*(n)$-bundle.*

By Lemma 1, we also obtain

Corollary. *If $n\geqq 5$, then τ has a connection as an $H_0(n)$-bundle.*

Note 1. We can show, any $t(x_0, x_1)$ which satisfies the conditions of Lemma 2, is written

$$t(x_0, x_1)\,|\,U=q'_U(x_0, x_1)^{-1}t_U(x_0, x_1),$$

where $t_U(x_0, x_1)$ is given by ($\sharp$), locally. Hence we may consider a connection of a topological manifold is an infinitesimal parallel displacement as the classical case.

Note 2. τ also has a connection as an $H_0(n)$-bundle if $n=1$ or 2. Similarly, for infinite dimensional manifolds, we obtain same result.

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