118. Note on Splitting Length of Abelian Groups

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Let A be an abelian group. Let tA denote the torsion part of A and A^n denote $A\otimes\cdots\otimes A$, n times. The splitting length $l(A)$ of an abelian group A is the least positive integer n, such that A^n splits.

In [3] J. M. Irwin, S. A. Khabbaz, and G. Rayna gave the definition of the splitting length of abelian groups as above and they proved that A^n splits if and only if $n\geq l(A)$ when tA is p-primary. We extend this result to abelian groups without p-primarity of the torsion part. First we have the next lemma for abelian groups A and B.

Lemma. *If $A\otimes B$ splits, then $A/tA\otimes B$ also splits.*

From the pure-exact sequence

$$0\to tB\to B\to B/tB\to 0$$

we obtain the exact sequence

$$0\to A/tA\otimes tB\to A/tA\otimes B\to A/tA\otimes B/tB\to 0$$

Let f be the natural homomorphism from A onto A/tA and i be the identity map of B. Then we have the following commutative diagram ([2], p. 33).

$$\begin{array}{lccccc} E: 0\to & t(A\otimes B) & \to & A\otimes B & \to & A/tA\otimes B/tB\to 0 \\ & \downarrow f\otimes i & & \downarrow f\otimes i & & \| \\ E': 0\to & A/tA\otimes tB & \to & A/tA\otimes B & \to & A/tA\otimes B/tB\to 0 \end{array}$$

Thus $E'=(f\otimes i)E$. Since $f\otimes i$ induces a homomorphism from $\mathrm{Ext}(A/tA\otimes B/tB, t(A\otimes B))$ into $\mathrm{Ext}(A/tA\otimes B/tB, A/tA\otimes tB)$ and E is splitting, E' is also splitting ([1], section 50). Moreover $A/tA\otimes tB$ is the torsion part of $A/tA\otimes B$. Thus $A/tA\otimes B$ splits.

Now we prove our theorem.

Theorem. *A^n splits if and only if $n\geq l(A)$.*

Suppose A^m splits. Then $A^{m-1}/tA^{m-1}\otimes A$ splits from the above lemma. Since $A^{m-1}/tA^{m-1}\cong(A/tA)^{m-1}$, $(A/tA)^{m-1}\otimes A$ splits. Then $(A/tA)^m\otimes A$ splits because A/tA is torsion-free. Suppose $A^m=T\oplus F$, where T is torsion and F is torsion-free. Then $A^{m+1}=(T\oplus F)\otimes A$ $=T\otimes A\oplus F\otimes A$.

Since $F\cong(A/tA)^m$, $F\otimes A$ splits. Considering $T\otimes A$ is torsion, A^{m+1} also splits. This concludes our proof.

References

[1] L. Fuchs: Infinite Abelian Groups. Academic Press (1970).
[2] P. A. Griffith: Infinite Abelian Group Theory. The University of Chicago Press (1970).
[3] J. M. Irwin, S. A. Khabbaz, and G. Rayna: The role of the tensor product in the splitting of abelian groups. Journal of Algebra, **14**, 423–442 (1970).