137. *Determination of $\tilde{K}_O(X)$ by $\tilde{K}_{SO}(X)$ for 4-Dimensional CW-Complexes*

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0. For a connected finite 4-dimensional CW-complex X we denote the group of stable vector bundles over X by $\tilde{K}_O(X)$, and the group of orientable stable vector bundles over X by $\tilde{K}_{SO}(X)$. In the previous paper [2] S. Sasao and the author determined the group structures of $\tilde{K}_{SO}(X)$ by cohomology rings. In this note we shall determine the relation between $\tilde{K}_O(X)$ and $\tilde{K}_{SO}(X)$. Our results include that $\tilde{K}_O(X) \cong \tilde{K}_{SO}(X) + H^1(X; Z_2)$ if and only if $Sq^1H^1(X; Z_2) = 0$. The author wishes to thank Professor S. Sasao for his valuable suggestions.

1. We can easily prove the following

Proposition 1. *The sequence*

$$0 \longrightarrow \tilde{K}_{SO}(X) \xrightarrow{i} \tilde{K}_O(X) \xrightarrow{W_1} H^1(X; Z_2) \longrightarrow 0$$

is exact, where i is a map which forgets the orientation and W_1 maps each class $[\xi]$ to the first Whitney class $W_1(\xi)$ of a bundle ξ which represents $[\xi]$.

This proposition shows that $\tilde{K}_O(X)$ is an element of $EXT(H^1(X; Z_2), \tilde{K}_{SO}(X))$. So we investigate this group.

Proposition 2. *There exists an isomorphism*

$$\varphi : EXT(H^1(X; Z_2), \tilde{K}_{SO}(X) \longrightarrow \sum_{i=1}^{r} (\tilde{K}_{SO}(X)/2\tilde{K}_{SO}(X))_i$$

where $r = \dim H^1(X; Z_2)$.

Proof. We assume that $H^1(X; Z_2) \cong \sum_{i=1}^{r} Z_2[\alpha_i]$, where [ ] denotes the generator. Consider the follwing exact sequence

$$0 \longrightarrow H \xrightarrow{i} F \xrightarrow{j} H^1(X; Z_2) \longrightarrow 0$$

where F is a free abelian group generated by $\{f_i\}$ such that $j(f_i) = \alpha_i$. By $\{h_i\}$ we denote generators of H corresponding to $\{2f_i\}$ via i. Then we know that there exists an isomorphism

$\rho : EXT(H^1(X; Z_2), \tilde{K}_{SO}(X)) \rightarrow HOM(H, \tilde{K}_{SO}(X))/\text{image } HOM(F, \tilde{K}_{SO}(X))$ defined as follows. For an exact sequence

$$0 \longrightarrow \tilde{K}_{SO}(X) \longrightarrow G \longrightarrow H^1(X; Z_2) \longrightarrow 0 ,$$

we take a set $\{g_i\}$ of elements of G going to $\{\alpha_i\}$. And we take a set $\{\gamma_i\}$ of elements of $\tilde{K}_{SO}(X)$ going to $\{2g_i\}$. Now we put $\rho(G)(h_i) = \gamma_i$ then $\rho(G)$ is uniquely defined as an element of $HOM(H, \tilde{K}_{SO}(X))/2HOM(H, \tilde{K}_{SO}(X)) \cong HOM(H, \tilde{K}_{SO}(X))/\text{image } HOM(F, \tilde{K}_{SO}(X))$. Let $p : \tilde{K}_{SO}(X) \rightarrow \tilde{K}_{SO}(X)$

$/2\tilde{K}_{SO}(X)$ be a natural projection. For any element k in $HOM(H, \tilde{K}_{SO}(X))$, we put $\psi(k)=\sum_{i=1}^{r} pk(h_i)$. Then it is easy to show that the map

$$\psi: HOM(H, \tilde{K}_{SO}(X))/2HOM(H, \tilde{K}_{SO}(X)) \longrightarrow \sum_{i=1}^{r} (\tilde{K}_{SO}(X)/2\tilde{K}_{SO}(X))_i$$

is bijective. Now $\varphi=\psi\circ\rho$ is the required isomorphism.

Let $H^1(X; Z_2)\cong\sum_i Z_2[\alpha_i]$ where [ ] denotes the generator of the group, $f_{\alpha_i}: X\to RP^\infty$ be the characteristic map of α_i, and ξ_0 be the canonical line bundle over RP^∞. We take $\eta_i=f_{\alpha_i}^*(\xi_0)$ in $\tilde{K}_O(X)$ to be the induced bundle of ξ_0 by f_{α_i}. Then Proposition 2 shows that $\varphi(\tilde{K}_O(X)) =\sum_i [2\eta_i]$ where $[2\eta_i]$ in $\tilde{K}_{SO}(X)/2\tilde{K}_{SO}(X)$ is the class represented by $2\eta_i$.

2. Now we assume that $H^1(X; Z_2)\cong\sum_{i=1}^{r} Z_2[\alpha_i]$.

At first we suppose that $\alpha_i{}^2\neq 0$. Then we have that $W(\eta_i)=1+\alpha_i$ for $\eta_i=f_{\alpha_i}^*(\xi_0)$, where $W(\eta_i)$ is the total Whitney class of η_i. For any element η in $\tilde{K}_{SO}(X)$, we have that

$$\begin{aligned} W(2(\eta_i\oplus\eta)) &= W(\eta_i\oplus\eta)^2 \\ &=(W(\eta_i)W(\eta))^2 \\ &=(1+\alpha_i)^2(1+W_2(\eta)+\cdots)^2 \\ &=1+\alpha_i^2+W_2(\eta)^2+\cdots \\ &\neq 1. \end{aligned}$$

Hence the bundle $2(\eta_i\oplus\eta)$ is non-trivial for any η in $\tilde{K}_{SO}(X)$. Thus we proved that if $\alpha_i^2\neq 0$, then $[2\eta_i]$ is non-zero in $\tilde{K}_{SO}(X)/2\tilde{K}_{SO}(X)$.

Secondly we suppose that $\alpha_i^2=0$. Consider the following commutative diagram;

$$\begin{array}{ccc} \eta_i \in \tilde{K}_O(X) & \xleftarrow{f_{\alpha_i}^{!}} & \tilde{K}_O(RP^\infty) \ni \xi_0 \\ c\big\downarrow & & c\big\downarrow \\ \eta_i' \in \tilde{K}(X) & \xleftarrow{f_{\alpha_i}^*} & \tilde{K}(RP^\infty) \ni \xi_0' \\ r\big\downarrow & & r\big\downarrow \\ 2\eta_i=\eta_i'' \in \tilde{K}_{SO}(X) & \xleftarrow{f_{\alpha_i}^{!}} & \tilde{K}_{SO}(PR^\infty) \ni \xi_0''=2\xi_0. \end{array}$$

Here $c: \tilde{K}_O(X)\to\tilde{K}(X)$ is the complexification and $r: \tilde{K}(X)\to\tilde{K}_{SO}(X)$ is the rearization. Then we have that $\eta_i''=rc(\eta_i)=2\eta_i$. The mod 2 reduction of the first Chern class of the bundle η_i' is as follows.

$$C_1(\eta_i')_2=W_2(\eta_i'')=W_2(2\eta_i)=\alpha_i^2=0.$$

So there exists an element γ_i in $H^2(X; Z)$ such that $C_1(\eta_i')=2\gamma_i$. Let $g_{\gamma_i}: X\to CP^\infty$ be the characteristic map of γ_i, and ζ_0 be the canonical complex line bundle over CP^∞. If we put $\theta_i=g_{\gamma_i}^*(\zeta_0)$ in $\tilde{K}(X)$, we have that $C_1(\theta_i)=\gamma_i$. So we get the equations that

$$C_1(\theta_i^2)=2\gamma_i=C_1(\eta_i').$$

As the bundles θ_i^2 and η_i' are complex line bundles over X, we have that $\theta_i^2=\eta_i'$. And the first Chern class of $g_{\gamma_i}^*(\zeta_0^2)$ is as follows.

$$C_1(g^*_{\gamma_i}(\zeta_0^2))=g^*_{\gamma_i}(C_1(\eta_0^2))=g^*_{\gamma_i}(2C_1(\zeta_0))=2C_1(g^*_{\gamma_i}(\zeta_0))$$
$$=2C_1(\theta_i)=2\gamma_i.$$

So we have that $g^*_{\gamma_i}(\zeta_0^2)=\eta'_i=\theta_i^2$.

If we assume that $\dim X=4$, we may use CP^2 for the classifying space CP^∞ in the above. So we have the following commutative diagram.

$$\begin{array}{ccc} \theta_i^2=\ \eta'_i\in\tilde{K}(X) & \xleftarrow{g^*_{\gamma_i}} & \tilde{K}(CP^2)\ni\zeta_0^2 \\ \downarrow r & & r\downarrow \\ 2\eta_i=\eta''_i\in\tilde{K}_{SO}(X) & \xleftarrow{g^*_{\gamma_i}} & \tilde{K}_{SO}(CP^2)\ni r(\zeta_0^2). \end{array}$$

We will prove that $r(\zeta_0^2)$ can be divided by 2 in $\tilde{K}_{SO}(CP^2)$. According to J. F. Adams [1], we have that $\tilde{K}(CP^2)\cong Z[\mu]+Z[\mu^2]$, where $\mu=\zeta_0-1$. As the complex line bundle ζ_0 equals to μ, we have that the element ζ_0^2 in $\tilde{K}(CP^2)$ comes from $\tilde{K}(S^4)$ in the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{K}(S^4) & \longrightarrow & \tilde{K}(CP^2) & \longrightarrow & \tilde{K}(S^2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{K}_{SO}(S^4) & \longrightarrow & \tilde{K}_{SO}(CP^2) & \longrightarrow & \tilde{K}_{SO}(S^2) & \longrightarrow & 0. \end{array}$$

Commutativity of the above diagram shows that $r(\zeta_0^2)$ is divisible by 2 in $\tilde{K}_{SO}(CP^2)$. Thus we proved that $[2\eta_i]=0$ in $\tilde{K}_{SO}(X)/2\tilde{K}_{SO}(X)$ if $\alpha_i^2=0$.

Summarizing the above, we have

Theorem. *Let X be a connected finite 4-dimensional CW-complex whose first cohomology group $H^1(X;Z_2)\cong\sum_{i=1}^r Z_2[\alpha_i]$. In*

$$EXT(H^1(X;Z_2),\tilde{K}_{SO}(X))\cong\sum_{i=1}^r(\tilde{K}_{SO}(X)/2\tilde{K}_{SO}(X))_i,$$

the direct summand $(\tilde{K}_{SO}(X)/2\tilde{K}_{SO}(X))_i$ of $\tilde{K}_O(X)$ corresponding to α_i is zero if and only if $\alpha_i^2=0$.

Remark. This theorem is valid for $\dim X\leqq 7$.

3. In this section we give an application. Let X be a connected finite 4-dimensional CW-complex. The results of [2] are the following:

If we represent cohomology groups of X so that they satisfy the following properties i)–ii).

$$H^2(X;\,Z_2)\cong\sum_{i=0}\sum_{j=1}^{s_i}Z_2[x_{ij}]+\sum_{k=1}^{s}Z_2[x_k].$$

$$H^4(X;\,Z_2)\cong\sum_{i=1}^{r_0}Z_2[\tilde{y}_i]+\sum_{i=1}\sum_{j=1}^{r_i}Z_2[\tilde{z}_{ij}].$$

$$H^4(X;\,Z)\cong\sum_{i=1}^{r_0}Z[y_i]+\sum_{i=1}\sum_{j=1}^{r_i}Z_{2^i}[z_{ij}]$$
$$+\sum_{p:\text{ odd prime}}\sum_{i=1}\sum_{j=1}^{t_i}Z_{p^i}[v_{pij}].$$

i) $x_k^2=0$, $x_{0j}^2=\tilde{y}_j$ for $1\leqq j\leqq s_0$, and $x_{ij}^2=\tilde{z}_{ij}$ for $1\leqq i$ and $1\leqq j\leqq s_i$.

ii) $\tilde{y}_i=i_1(y_i)$, and $\tilde{z}_{ij}=i_1(z_{ij})$, where i_1: $H^4(X;\,Z)\to H^4(X;\,Z_2)$.

Then we have that

$$\tilde{K}_{SO}(X) \cong \sum_1^s Z_2 + \sum_{i=1} \sum_{j=1}^{s_i} Z_{2i+1} + \sum_1^{r_0} Z$$
$$+ \sum_{i=1} \sum_{j=s_i+1}^{r_i} Z_{2i} + \sum_{p:\text{ odd prime}} \sum_{i=1} \sum_{j=1}^{t_i} Z_{p^i}.$$

Let us assume that $H^1(X; Z_2) \cong Z_2[\alpha]$, and $\alpha^2 \neq 0$.

a) If $\alpha^4 = 0$, there exists an element x_k in $H^2(X; Z_2)$ such that $\alpha^2 = x_k$. So we have that $2f_\alpha^*(\xi_0)$ is equivalent to the generator of order 2 in $\tilde{K}_{SO}(X)$ (which is η_k in [2]). Thus we have that $\tilde{K}_O(X)$ is isomorphic to the group replacing a summand Z_2 with Z_4 in $\tilde{K}_{SO}(X)$.

b) If $\alpha^4 \neq 0$, we have that $\delta(\alpha)^2 \neq 0$ where $\delta: H^1(X; Z_2) \rightarrow H^2(X: Z)$ is the connecting homomorphism. The fact that $2\delta(\alpha) = 0$ shows that there exists an element z_{1j} in $H^4(X; Z)$ such that $\delta(\alpha)^2 = z_{1j}$. Thus we have that $2f_\alpha^*(\xi_0)$ is equivalent to the generator of order 4 in $\tilde{K}_{SO}(X)$ (which is η'_{1j} in [2]). Now $\tilde{K}_O(X)$ is isomorphic to the group replacing a summand Z_4 with Z_8 in $\tilde{K}_{SO}(X)$.

References

[1] J. F. Adams: Vector fields on spheres. Ann. of Math., **75**, 603–632 (1962).
[2] S. Sasao and Y. Ando: *KSO*-groups for 4-dimensional *CW*-complexes. Nagoya Math. J., **42**, 23–29 (1971).