169. *Some Dual Properties on Modules*

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As mutually dual notions on modules, we have projective module and injective module, quasi-projective module and quasi-injective module, minimal homomorphism and essential homomorphism etc. We know that some properties for ones have dual properties for others. But it is not necessarily always true. In this short note, we collect some properties on modules which has dual property. Some dual property of one can be easily proved by exchanging the notions to its duals. We assume that R is any ring with unit element and every R-module is unitary R-right (or left) module. For R-module M and its R-sub module N, the symbol $N \hookrightarrow M$ denotes the inclusion map $x \mapsto x$, and $M \xrightarrow{-} M/N$ denotes the canonical epimorphism $x \mapsto \bar{x}$.

Property 1. *For any R-module* M, $\mathfrak{A}=\{f \in End_R(M)\,;\, Ker\, f \hookrightarrow M$ *is essential*$\}$ *is a two sided ideal of* $End_R(M)$.

Proof. For any $f, f' \in \mathfrak{A}$, $Ker\, f \cap Ker\, f' \subset Ker\,(f+f') \subset M$, therefore $f+f' \in \mathfrak{A}$. For $f \in \mathfrak{A}$ and $g \in End_R(M)$, $M \supset Ker\,(gf)\; Ker\, f$, hence $gf \in \mathfrak{A}$. On the other hand, for any R-submodule U of M $Ker\, f \cap g(U) \subset g\,(Ker\,(fg) \cap U)$, hence if $Ker\,(fg) \cap U=0$ then $Ker\, f \cap g(U)=0$ and so $g(U)=0$. Therefore $U \subset Ker\, g \subset Ker\,(f \cdot g)$ and $U \subset Ker\,(f \cdot g) \cap U=0$. Accordingly $f \cdot g \in \mathfrak{A}$.

Property 1*) (Harada and Kambara [1], Lemma 1). *For any R-module* $M, \mathfrak{A}^*=\{f \in End_R(M)\,;\, M \xrightarrow{-} Coker\, f$ *is minimal*$\}$ *is a twosided ideal of* R.

Property 2 (Faith [2], p. 44, Theorem 1). *If* M *is quasi-injective R-module, then is Jacobson radical of* $End_R(M)$.

Property 2*). *If* M *is quasi-projective R-module, then* $\mathfrak{A}^*$ *is Jacobson radical of* $End_R(M)$.

Proof. Let J be the radical of $End_R(M)$. Then $J \supset \mathfrak{A}^*$ is obvious. Because, for any f in $\mathfrak{A}^*$, $Im\, f+Im\,(1-f)=M$, therefore $Im\,(1-f)=M$, i.e. $1-f: M \to M$ is R-epimorphism. By quasi-projective of M, $1-f$ has right inverse h in $End_R(M)$, i.e.

$$\begin{tikzcd} & M \arrow[dl, "h"'] \arrow[d, "I"] & \\ M \arrow[r, "1-f"'] & M \arrow[r] & 0 \end{tikzcd}$$

is commutative. Therefore $f \in J$. Conversely, let f be in J, and U

any R-sub module of M. If $Im\, f+U=M$, then there exists g in $End_R(M)$ such that

$$\begin{tikzcd} & M \arrow[dl, "g"'] \arrow[d, "h"] & \\ M \arrow[r, "\bar{f}"] & M/U \arrow[r] & 0 \end{tikzcd}$$

is commutative, where $h: M \to M/U$ is the canonical homomorphism, and $\bar{f}: M \to M/U$; $m \mapsto \overline{f(m)}$. That is, $f \circ g(x) \equiv x \pmod U$, i.e. $(fg-1)(x) \in U$ for all xM. Since $f \circ g - 1$ is inversible in $End_R(M)$, $M = f \circ g - 1(M)$ and so $M = f \circ g - 1(M) = U$ accordingly $f \in \mathfrak{A}^*$. We have $\mathfrak{A}^* = J$.

Property 3. *Let M and M' be R-modules and $f: M \to M'$ an R-homomorphism. If inclusion map $Ker\, f \subsetneq M$ is essential, then $Ker\, f$ contains the socle $S(M)$ $f \circ M$.*

Proof. If there exists a minimal R-sub module U of M, then we have $U \cap Ker\, f = 0$ or $= U$. If $U \cap Ker\, f = 0$, then by essential of $Ker\, f \subsetneq M$, we have $U = 0$. If $U \cap Ker\, f = U$ then $U \subset Ker\, f$, therefore, $S(M) \subset Ker\, f$.

Property 3*). *Let M' and M be R-modules, and $f: M' \to M$ an R-homomorphism. If the canonical map $M \xrightarrow{-} Coker\, f$ is minimal, then $Im\, f$ is contained in the Jacobson radical $J(M)$ of M.*

Definition. R-module M is called co-finitely generated R-module if for any family $\{N_\nu\}_{\nu \in I}$ of R-sub modules N_ν of M, $\bigcap_{\nu \in I} N_\nu = 0$ implies that there exists a finite number of N_ν's, say $N_{\nu_1}, N_{\nu_2}, \cdots N_{\nu_n}$, such that $N_{\nu_1} \cap N_{\nu_2} \cap N_{\nu_3} \cap \cdots \cap N_{\nu_n} = 0$.

Property 4. *Let M be a co-finitely generated R-module, M' an arbitrary R-module, and $f: M \to M'$ any R-homomorphism. Then the inclusion map $Ker\, f \subsetneq M$ is essential if and only if $Ker\, f \supset S(M)$.*

Proof. The part of "only if" is in 3. Let $Ker\, f \supset S(M)$. By co-finitely generated of M, any R-sub module $N \neq 0$ of M contains a minimal R-sub module $U \neq 0$. Therefore $N \cap Ker\, f \supset U \cap S(M) = U \neq 0$ i.e. $Ker\, f \subsetneq M$ is essential.

Property 4*). *Let M be finitely generated R-module, M' an arbitrary R-module, and $f: M' \to M$ any R-homomorphism. Then the canonical map $M \xrightarrow{-} Coker\, f$ is minimal if and only if $Im\, f \subset J(M)$.*

Property 5. *Let M be quasi-injective R-module. If the inclusion map $S(M) \subsetneq M$ is essential, then $Im\,(Hom_R(M/S(M), M) \xrightarrow{j^*} Hom_R(M, M))$ is the Jacobson radical of $End_R(M)$ for canonical map $j: M \xrightarrow{-} M/S(M)$.*

Proof. Let $f \in Im\,(Hom_R(M/S(M), M) \xrightarrow{j^*} Hom_R(M, M))$. Then $f \in End_R(M)$ and $Ker\, f \supset S(M)$. By the assumption, $Ker\, f \subsetneq M$ is essential and by 2, f is in Jacobson radical of $End_R(M)$. Conversely, if f is in Jacobson radical of $End_R(M)$, then by 2 $Ker\,f \subsetneq M$ is essential and by 3 $Ker\,f \supset S(M)$, therefore $S(M) \subsetneq M$ is essential.

Property 5*). *Let M be quasi-projective R-module. If the canonical map $M \xrightarrow{-} M/J(M)$ is minimal, then $Im\,(Hom_R\,(M, J(M)) \xrightarrow{i^*} Hom_R(M, M))$ is the Jacobson radical of $End_R(M)$ for $i: J(M) \subsetneq M$.*

Property 6. *Let M be injective R-module. Then, $S(M) \subsetneq M$ is essential if and only if $Im\,(Hom_R\,(M/S(M), M) \xrightarrow{j^*} Hom_R\,(M, M))$ is the Jacobson radical of $End_R\,(M)$.*

Proof. The part of "only if" is in 5. If $Im\,(Hom_R\,(M/S(M), M) \rightarrow Hom_R\,(M, M))$ is the radical of $End_R\,(M)$, then for any R-sub module $N \neq 0$ of M we can show $S(M) \cap N \neq 0$. If $S(M) \cap N = 0$, then there exists R-homomorphism $h: M/S(M) \rightarrow M$ such that

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "-"] \arrow[d, "\cap"'] & M/S(M) \arrow[dl, "h"] \\ & M & \end{tikzcd}$$

is commutative. Let $h' = (M \xrightarrow{-} M/S(M) \xrightarrow{h} M)$. Then $h' \in Im\,(Hom_R\,(M/S(M), M) \rightarrow Hom_R\,(M, M)) =$ radical of $End_R\,(M)$, and $Ker\,h' \cap N = 0$. Therefore, $Ker\,h' \subsetneq M$ is essential, and so $N = 0$, it is a contradiction.

Property 6*) (Harada and Kambara [1], Proposition 1). *Let M be projective R-module. Then, $M \xrightarrow{-} M/J(M)$ is minimal if and only if $Im\,(Hom_R\,(M, J(M)) \xrightarrow{i^*} Hom_R\,(M, M))$ the Jacobson radical of $End_R\,(M)$.*

Property 7. *Let M be quasi-injective R-module. If for every R-module $N \neq 0$ of M, the inclusion map $N \subsetneq M$ is essential, then $End_R\,(M)$ is local ring.*

Proof. Let f in $End_R\,(M)$. If f is not in the radical of $End_R\,(M)$, then $Ker\,f = 0$ by 2. Since M is quasi-injective, there exists g in $End_R\,(M)$ such that

$$\begin{tikzcd} 0 \arrow[r] & f(M) \arrow[r] \arrow[d, "f^{-1}"] & M \arrow[dl, "g"] \\ & M & \end{tikzcd}$$

is commutative, i.e. $x = f^{-1}f(x) = gf(x)$ for all $x \in M$. Therefore $g \circ f = 1$, and $g \notin J =$ radical of $End_R\,(M)$, therefore f is unit in $End_R\,(M)$. Accordingly, $End_R\,(M)$ is local ring.

Property 7*). *Let M be quasi-projective R-module. If for every R-sub module $N \neq M$ of M the canonical map $M \xrightarrow{-} M/N$ is minimal, then $End_R\,(M)$ is local ring.*

Property 8. *Let M be co-finitely generated quasi-injective R-*

module. If M has unique minimal R-sub module U, then $End_R(M)$ is local ring.

Property 8*). *Let M be finitely generated quasi-projective R-module. If M has unique maximal R-sub module U, then $End_R(M)$ is local ring.*

References

[1] M. Harada and H. Kambara: On categories of projective modules (to appear).

[2] C. Faith: Lecture on injective module and quotient rings. Springer-Verlag, Lecture Note in Maths. 49 (1967).

[3] H. Bass: Finitistic dimension and homological generalization of semi-primary ring. Trans. A.M.S., **95**, 446–488 (1960).