4. On a Nuclear Function Space on a Topological Space

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In this paper, we construct the nuclear function space on a topological space X with a measure μ such that $\mu(K)<\infty$ for every compact subset K of X.

The construction can, of course, be adopted for locally compact groups with a left or right Haar measure, and so the function spaces defined in the present paper include those due to Pietsh [1].

For nuclear spaces and the related notions, see [2]. Throughout this paper we denote by $C(X)$ the space of all continuous functions on a topological space X.

Definition. Let X be a topological space with a measure μ such that $\mu(K)<\infty$ for every compact subset K of X. A subspace $N(X)$ of $C(X)$ is said to have the property (A) if for every compact subset K there exist a compact subset $H\supset K$ and a positive real number ρ such that

$$\sup\{|\phi(x)|\,;\,x\in K\}\leqq\rho\left\{\int_H|\phi(x)|^2d\mu(x)\right\}^{1/2}$$

for every $\phi\in N(X)$.

Theorem 1. *Let X be a topological space with a measure μ such that $\mu(K)<\infty$ for every compact subset K of X. Let $N(X)$ be a subspace of $C(X)$ with the property (A).*

Define the topology of $N(X)$ by the system of seminorms

$$\|\phi\|_K=\left\{\int_K|\phi(x)|^2d\mu(x)\right\}^{1/2},\qquad K\in\mathfrak{S}$$

where $\mathfrak{S}$ is the family of all compact subsets of X. Then the locally convex space $N(X)$ is a nuclear space.

Proof. Let K be a member of $\mathfrak{S}$ and let $\wedge$ be a canonical mapping of $N(X)$ into $N(X)/\{\phi\,;\,\|\phi\|_K=0\}$. We denote by H_K the completion of the quotient space $N(X)/\{\phi\,;\,\|\phi\|_K=0\}$ with respect to the quotient norm. Then it is clear that H_K is a Hilbert space.

In order to show that $N(X)$ is a nuclear space, it suffices to prove that for any compact subset K there exists a compact subset $T\supset K$ such that the canonical mapping $H_T\rightarrow H_K$ is of Hilbert-Schmidt type. By the assumption, we have the inequality

$$\sup\{|\phi(x)|\,;\,x\in K\}\leqq\rho\left\{\int_T|\phi(x)|^2d\mu(x)\right\}^{1/2}$$

for every $\phi\in N(X)$.

Now let

$$\langle\hat{\phi}, \delta_x\rangle=\phi(x) \quad \text{for every} \quad x\in K \quad \text{and every} \quad \phi\in N(X).$$

Then δ_x is a continuous linear form defined on a dense subspace of H_T. The continuous linear form δ_x can be extended continuously to the whole space H_T. We denote again by δ_x the extended continuous linear form. Consequently, we have for every $x\in K$

$$|\langle\hat{\phi}, \delta_x\rangle|=|\phi(x)|\leqq\rho\left\{\int_T|\phi(x)|^2 d\mu(x)\right\}^{1/2},$$

and so

$$\|\delta_x\|'_T\leqq\rho$$

where $\| \ \|'$ denotes the norm of the dual of H_T.

Let $\{\hat{\phi}_i\}$ be a complete orthonormal system of H_T and let $\{\phi_i\}$ be the corresponding elements of $N(X)$. By the Parseval's equality, we have

$$\sum_{i=1}^{\infty}|\phi_i(x)|^2=\sum_{i=1}^{\infty}|\langle\hat{\phi}_i, \delta_x\rangle|^2=\|\delta_x\|'^2_T\leqq\rho^2 \quad \text{for every} \quad x\in K.$$

Hence

$$\sum_{i=1}^{\infty}\|\hat{\phi}_i\|_K^2=\sum_{i=1}^{\infty}\int_K|\phi_i(x)|^2 d\mu(x)\leqq\rho^2\mu(K).$$

Therefore the canonical mapping $H_T\rightarrow H_K$ is of Hilbert-Schmidt type. This completes the proof.

In particular, if topological space X is σ-compact; then there exists a sequence $\{X_n\}$ of compact subsets of X such that $X=\bigcup_{n=1}^{\infty}X_n$ and $X_n\subset X_{n+1}$ for all $n\in N$; and consequently $N(X)$ is a (complete) metrizable nuclear space.

The next theorem gives a partial converse of Theorem 1.

Theorem 2. *Let X be a σ-compact topological space and let $\{X_n\}$ be a sequence of compact subsets of X such that $X=\bigcup_{n=1}^{\infty}X_n$ and $X_n\subset X_{n+1}$ for all $n\in N$. Moreover let $M(X)$ be a subspace of $C(X)$ with the locally convex topology defined by the system of countable norms*

$$\|\phi\|_n=\sup\{|\phi(x)|;\ x\in X_n\}.$$

If $M(X)$ is a nuclear space, then there exists a positive Radon measure μ on X such that $M(X)$ has the property (A).

Proof. Since $M(X)$ is a nuclear space, for any positive integer n there exists $m>n$ such that the canonical mapping $H_m\rightarrow H_n$ is a nuclear mapping where $H_i(i=1, 2, \cdots)$ is the completion of the quotient space $M(X)/\{\phi;\ \|\phi\|_i=0\}$ with respect to the quotient norm. It follows that we can find a sequence $\{a_i\}$ of continuous linear forms on H_m with $\sum_{i=1}^{\infty}\|a_i\|'_m<\infty$ and a sequence $\{\phi_i\}$ in H_n such that

$$\|\phi_i\|_n\leqq 1$$

and

$$\phi(x)=\sum_{i=1}^{\infty}<\phi, a_i>\phi_i(x) \quad \text{for every} \quad \phi\in H_m \quad \text{and for all} \quad x\in X_n.$$

Each continuous linear form a_i can be extended to continuous linear form μ_i on $C(X_m)$ with $\|a_i\|'_m=\|\mu_i\|'$ where $C(X_m)$ is the normed space of all continuous functions on X_m with the norm $\|\phi\|=\sup\{|\phi(x)|;\ x\in X_m\}$ and $\|\ \|'$ denotes the norm of the dual of $C(X_m)$.

Now we define a positive Radom measure μ_{X_m} on X_m by

$$\mu_{X_m}=\sum_{i=1}^{\infty}|\mu_i|.$$

Then we have

$$\begin{aligned}|\phi(x)|\leqq\sum_{i=1}^{\infty}|\langle\phi,a_i\rangle|&\leqq\sum_{i=1}^{\infty}\langle|\phi|,\mu_i\rangle\\&=\langle|\phi|,\mu_{X_m}\rangle\\&=\int_{X_m}|\phi(x)|\,d\mu_{X_m}(x)\end{aligned}$$

for every $x\in X_n$.

Hence

$$\sup\{|\phi(x)|;\ x\in X_n\}\leqq\int_{X_m}|\phi(x)|\,d\mu_{X_m}(x)\quad\text{for all}\quad\phi\in M(X).$$

Let us define a positive Radom measure μ on X by

$$\int_X|\phi(x)|\,d\mu(x)=\sum_{n=1}^{\infty}2^{-n}(\mu_{X_n}(X_n))^{-1}\int_{X_n}|\phi(x)|\,d\mu_{X_n}(x).$$

Then we have, for each $\phi\in M(X)$

$$\sup\{|\phi(x)|;\ x\in X_n\}\leqq\rho'\int_{X_m}|\phi(x)|\,d\mu(x)$$

where $\rho'=2^m\mu_{X_m}(X_m)$.

On the other hand, as is well known

$$\left\{\int_{X_m}|\phi(x)|^p\,d\mu(x)\right\}^{1/p}\leqq\mu(X_m)^{1/p-1/q}\left\{\int_{X_m}|\phi(x)|^q\,d\mu(x)\right\}^{1/q}$$

for all integers p, q such that $q\geqq p\geqq 1$.

Thus we have the required inequality

$$\sup\{|\phi(x)|;\ x\in X_n\}\leqq\rho\int_{X_m}|\phi(x)|^2\,d\mu(x).$$

This completes the proof.

References

[1] Pietsch, A.: Nukleare Funktionenräume. Math. Nachr., **33**, 377–384 (1967).
[2] ——: Nukleare lokalkonvexe Räume. Berlin (1965).