3. On Integral Inequalities Related with a Certain Nonlinear Differential Equation

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As is shown in [3], the following nonlinear differential equation:

$$(1)\qquad nh(1-h^2)\frac{d^2h}{dt^2}+\left(\frac{dh}{dt}\right)^2+(1-h^2)(nh^2-1)=0,$$

where n is any integer $\geqq 2$, is the equation for the support function $h(t)$ of a plane curve in the unit disk: $u^2+v^2<1$, with respect to the tangent direction angle t, which is related with a minimal hypersurface in the $(n+1)$-dimensional unit sphere. Any solution $h(t)$ of (1) such that $h^2+\left(\frac{dh}{dt}\right)^2<1$ is periodic and its period T is given by the improper integral:

$$(2)\qquad T(C)=2\int_{a_0}^{a_1}\frac{dh}{\sqrt{1-h^2-C\left(\frac{1}{h^2}-1\right)^{1/n}}},$$

where

$$C=(a_0^2)^{1/n}(1-a_0^2)^{1-(1/n)}=(a_1^2)^{1/n}(1-a_1^2)^{1-(1/n)}$$
$$\left(0<a_0<\frac{1}{\sqrt{n}}<a_1\right)$$

is the integral constant of (1). Regarding the function $T(C)$, $0<C<A=(1/n)^{1/n}(1-(1/n))^{1-(1/n)}$, the following is known in [3]:

(i) *$T(C)$ is differentiable and $T(C)>\pi$,*

(ii) $\lim_{C\to 0}T(C)=\pi$ *and* $\lim_{C\to A}T(C)=\sqrt{2}\,\pi$.

Putting $h^2=x$, $a_0^2=x_0$, $a_1^2=x_1$ and $1/n=\alpha$, (2) can be written as

$$(3)\qquad T(C)=\int_{x_0}^{x_1}\frac{dx}{\sqrt{x(1-x)-C\psi(1-x)}},$$

where

$$(4)\qquad \psi(x)=x^{\alpha}(1-x)^{1-\alpha}\qquad \text{on } 0<x<1$$

and

$$(5)\qquad C=\psi(x_0)=\psi(x_1),\qquad 0<x_0<\alpha<x_1<1,$$

$$(6)\qquad 0<C<A=\psi(\alpha).$$

Now, suppose that α is any real number such that

$$(7)\qquad 0<\alpha\leqq 1/2$$

and consider as the function $T(C)$ is defined by the right hand side of (3) on the interval (6). Then, we have

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*) Dedicated to Professor Yoshie Katsurada on her 60th birth day.

Theorem. *For the integral $T(C)$, we have the following inequality:*

$$T(C) < \left(\frac{1}{\sqrt{2}} + \sqrt{1-\alpha}\right)\pi.$$

Proof. We have easily

$$\psi(x)\psi(1-x) = x(1-x), \tag{8}$$

$$\frac{d\psi(x)}{dx} = \frac{\alpha - x}{x(1-x)}\psi(x) \tag{9}$$

and

$$\frac{d\psi(1-x)}{dx} = \frac{1-\alpha-x}{x(1-x)}\psi(1-x). \tag{10}$$

$\psi(x)$ is monotone increasing on $0<x<\alpha$ and monotone decreasing on $\alpha<x<1$. Let $X_0(u)$ and $X_1(u)$ be the inverse functions of $u=\psi(x)$ on $0<x<\alpha$ and $\alpha<x<1$ respectively. Thus, changing the integral parameter x in (3) to $u=\psi(x)$ and using (8) and (9), $T(C)$ can be written as

$$\begin{aligned} T(C) &= \int_{x_0}^{\alpha} \frac{dx}{\sqrt{x(1-x) - C\psi(1-x)}} + \int_{\alpha}^{x_1} \frac{dx}{\sqrt{x(1-x) - C\psi(1-x)}} \\ &= \int_C^A \frac{\sqrt{X_0(u)(1-X_0(u))}}{(\alpha - X_0(u))\sqrt{u(u-C)}}du \\ &\quad + \int_A^C \frac{\sqrt{X_1(u)(1-X_1(u))}}{(\alpha - X_1(u))\sqrt{u(u-C)}}du \\ &= \int_C^A \frac{\sqrt{X_0(u)(1-X_0(u))(A-u)}}{(\alpha - X_0(u))\sqrt{u}} \cdot \frac{du}{\sqrt{(A-u)(u-C)}} \\ &\quad + \int_C^A \frac{\sqrt{X_1(u)(1-X_1(u))(A-u)}}{(X_1(u)-\alpha)\sqrt{u}} \cdot \frac{du}{\sqrt{(A-u)(u-C)}}. \end{aligned}$$

Now, we assume that

$$\frac{\sqrt{X_i(u)(1-X_i(u))(A-u)}}{|\alpha - X_i(u)|\sqrt{u}} \leqq \lambda_i \tag{11}$$

for $C \leqq u < A$, $i=0, 1$. Then, we have

$$T(C) < (\lambda_0 + \lambda_1)\int_C^A \frac{du}{\sqrt{(A-u)(u-C)}} = (\lambda_0+\lambda_1)\pi. \tag{12}$$

In the following, we shall show that we can take the values of λ_0 and λ_1 as

$$\lambda_0 = 1/\sqrt{2} \qquad \text{and} \qquad \lambda_1 = \sqrt{1-\alpha}.$$

The inequalities (11) are equivalent to

$$\frac{\sqrt{x(1-x)(A-\psi(x))}}{|\alpha - x|\sqrt{\psi(x)}} \leqq \lambda_i \tag{13}$$

for $x_0 \leqq x < \alpha$ and $\alpha < x \leqq x_1$ respectively. Setting $\lambda = \lambda_0$, λ_i, (13) is equivalent to

$$x(1-x)(A-\psi(x)) \leqq \lambda^2(\alpha-x)^2\psi(x),$$

that is

$$x(1-x)A \leqq \psi(x)[\lambda^2(\alpha-x)^2 + x(1-x)].$$

By (8), this inequality can be written as

$$A \leqq \frac{\lambda^2(\alpha-x)^2+x(1-x)}{\psi(1-x)} :=f_\lambda(x). \tag{14}$$

For this positive valued function $f_\lambda(x)$ on $0<x<1$ for any $\lambda>0$, we have

$$f_\lambda(\alpha)=A, \tag{15}$$

and

$$\frac{f_\lambda'}{f_\lambda}=\frac{-2\lambda^2(\alpha-x)+1-2x}{\lambda^2(\alpha-x)^2+x(1-x)}-\frac{1-\alpha-x}{x(1-x)}$$
$$=\frac{g_\lambda(x)}{x(1-x)[\lambda^2(\alpha-x)^2+x(1-x)]},$$

where

$$g_\lambda(x)=(\alpha-x)[-\lambda^2\alpha(1-\alpha)+(1-\lambda^2)x(1-x)]. \tag{16}$$

i) Case $\lambda=1/\sqrt{2}$. We have

$$g_\lambda(x)=-\frac{1}{2}(x-\alpha)^2(1-\alpha-x)$$

which shows that (14) holds on the interval $0<x<\alpha$, but not on any interval $(\alpha, x_1]$.

ii) Case $\lambda=\sqrt{1-\alpha}$. We have

$$g_\lambda(x)=(\alpha-x)[-\alpha(1-\alpha)^2+\alpha x(1-x)]$$
$$=\alpha(x-\alpha)[x^2-x+(1-\alpha)^2]$$

and

$$1-4(1-\alpha)^2 \leqq 1-4\left(1-\frac{1}{2}\right)^2=0$$

by (7), which shows that (14) holds on the interval $0<x<1$.

Thus, we have proved that (11) are true when we put $\lambda_0=1/\sqrt{2}$ and $\lambda_1=\sqrt{1-\alpha}$. Hence, we get from (12)

$$T(C)<\left(\frac{1}{\sqrt{2}}+\sqrt{1-\alpha}\right)\pi. \qquad \text{Q.E.D.}$$

Remark. The author wanted originally to have the inequality: $T(C)<2\pi$ from the standpoint of a geometrical problem and S. Furuya gave firstly an answer to it by proving the inequality: $T(C)<\sqrt{(n-1)/n} \times 2\pi$ in [1]. By means of a numerical analysis and observation on (1) done by M. Urabe, it is expected to have the inequality: $T(C)<\sqrt{2}\times\pi$ in [4].

References

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