2. A Note on Primitive Extensions of Rank 4 of Alternating Groups

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1. In [11] and [5], the primitive extensions of rank 3 of symmetric groups or alternating groups which act naturally have been determined. More generally, E. Bannai has determined the primitive extensions of rank 3 of 4-ply transitive permutation groups ([1]). In this note we consider the primitive extensions of rank 4 of alternating groups which act naturally. Before stating our result, we note the following. It gives an example of primitive and imprimitive groups of an arbitrary rank.

Let H_k be the symmetric group S_k or the alternating group A_k which act naturally on $\{1, 2, \cdots, k\}$ and let E_{k-1} be the elementary abelian group of order 2^{k-1}. Furthermore, let $a_1, a_2, \cdots, a_{k-1}$ be a minimal set of generators of E_{k-1} and put $a_k = a_1 a_2 \cdots a_{k-1}$. Every element σ of H_k induces an automorphism $\bar{\sigma}$ of E_{k-1} defined by

$$\bar{\sigma} = \begin{pmatrix} a_1 \, a_2 \, \cdots a_k \, a_1 a_2 & \cdots \\ a_{1\sigma} a_{2\sigma} \cdots a_{k\sigma} a_{1\sigma} a_{2\sigma} & \cdots \end{pmatrix}.$$

Thus H_k is identified with an automorphism group of E_{k-1}.
Construct the semidirect product $H_k \cdot E_{k-1}$ and let it act naturally on $\{H_k x \mid x \in E_{k-1}\}$, the set of right cosets of H_k. Then we have

Proposition (cf. 4 (iv) in Tsuzuku [11]). *For $n \geqq 2$, $H_{2n} \cdot E_{2n-1}$ is an imprimitive rank $n+1$ group of degree 2^{2n-1} with subdegrees $1, \binom{2n}{1}, \binom{2n}{2}, \cdots, \binom{2n}{n-1}, 1/2\binom{2n}{n}$, and $H_{2n+1} \cdot E_{2n}$ is a primitive rank $n+1$ group of degree 2^{2n} with subdegrees*

$$1, \binom{2n+1}{1}, \binom{2n+1}{2}, \cdots, \binom{2n+1}{n-1}, \binom{2n+1}{n}.$$

Theorem. *Let A_k be the alternating group of degree k. If A_k has a primitive extension G of rank 4, then $k=7$ and G is isomorphic to $A_7 \cdot E_6$.*

2. Outline of a proof of Theorem.

Notation.

S_k: The symmetric group of degree k.

A_k: The alternating group of degree k (on a set Δ_1).

G: A primitive extension of rank 4 of A_k on a set $\Omega = \{0, 1, 2, \cdots, k, \tilde{1}, \tilde{2}, \cdots, \tilde{l}, \bar{1}, \bar{2}, \cdots, \bar{m}\}$.

H: The stabilizer G_0 of a letter, say 0, of Ω. The non-trivial orbits of H are denoted by $\Delta=\{1, 2, \cdots, k\}$, $\Gamma=\{\tilde{1}, \tilde{2}, \cdots, \tilde{l}\}$ and $\Lambda=\{\bar{1}, \bar{2}, \cdots, \bar{m}\}$. By definition H acts faithfully on Δ and the permutation group (H, Δ) is isomorphic to (A_k, Δ_1).

L: The pointwise stabilizer $G_{0,\tilde{1}}$ of the subset $\{0, \tilde{1}\}$ of Ω. Clearly $[H: L]=l$.

M: The pointwise stabilizer $G_{0,\bar{1}}$ of the subset $\{0, \bar{1}\}$ of Ω. Clearly $[H: M]=m$.

For the stabilizer G_a of $a \in \Omega$, $\Delta(a)$, $\Gamma(a)$ and $\Lambda(a)$ denote the non-trivial orbits of G_a such that $\Delta(a)^g=\Delta(a^g)$, $\Gamma(a)^g=\Gamma(a^g)$ and $\Lambda(a)^g=\Lambda(a^g)$ for $g \in G$. In particular, $\Delta(0)=\Delta$, $\Gamma(0)=\Gamma$ and $\Lambda(0)=\Lambda$.

For a group Y and a subgroup X of Y,

1_X : The principal character of X.

1_X^Y : The character of Y induced by 1_X.

$1_{X|X}^Y$: The restriction to X of 1_X^Y.

The main idea of our proof is indebted to T. Tsuzuku and D. G. Higman ([11], [3], [4]). That is, we first determine the values of l and m or the structure of L and M as in [11] and [5], and next, except some cases, we show the non-existence of a primitive permutation group of rank 4 which has 1, k and such l, m as subdegrees, by using a general theory on rank 4 groups ([6]). The remaining cases are dealt with similarly, considering the structure of L.

It is easily seen that $k \geqq 4$ and so (H, Δ) is doubly transitive. Hence by a theorem of Manning [8] we may assume that $k<l$ and l is a divisor of $k(k-1)$. Further, from the proof of a theorem of Manning, we may assume that

$$|\Delta(0) \cap \Delta(1)|=0, \ |\Delta(0) \cap \Gamma(1)|=k-1 \quad \text{and} \quad |\Delta(0) \cap \Lambda(1)|=0. \qquad (*)$$

Step 1. Determining the value of l and the structure of L. As in [11] and [5] we have

Lemma 1. *If $k \geqq 9$, then we have either $L=S_1 \times S_1 \times A_{k-2}$ and $l=k(k-1)$ or $L=S_2 \times S_{k-2} \cap A_k$ and $l=k(k-1)/2$.*

Step 2. Determining the value of m and the structure of M. To do this, we need only determine the subgroups of A_k with indices $\leqq k(k-1)^2$ since we know that $[H: M]=m \leqq k(k-1)^2$ by Proposition 3.1 in [6]. A similar (but rather long) argument as in the proof of Lemma 1 yields

Lemma 2. *Let K be a subgroup of A_k with $[A_k: K] \leqq k(k-1)^2$.* Then, for $k>15$ K is one of the following groups.

(1) $S_1 \times S_1 \times S_1 \times A_{k-3}$ (in this case $[A_k: K]=k(k-1)(k-2)$)

(2) $S_1 \times S_1 \times A_{k-2}$ (in this case $[A_k: K]=k(k-1)$)

(3) $S_1 \times A_{k-1}$ (in this case $[A_k: K]=k$)

(4) $S_1 \times (S_2 \times S_{k-3} \cap A_{k-1})$ (in this case $[A_k: K]=k(k-1)(k-2)/2$)

(5) $A_3 \times A_{k-3}$ (in this case $[A_k : K] = k(k-1)(k-2)/3$)
(6) $S_2 \times S_{k-2} \cap A_k$ (in this case $[A_k : K] = k(k-1)/2$)
(7) $S_3 \times S_{k-3} \cap A_k$ (in this case $[A_k : K] = k(k-1)(k-2)/6$)
(8) $S_4 \times S_{k-4} \cap A_k$ (in this case $k \leqq 27$ and $[A_k : K] = \binom{k}{4}$)

Thus l and m are determined by k.

Step 3. For such l and m except some cases, regardless of the structures of L and M, [6] shows the non-existence of a primitive permutation group of rank 4 which has $1, k, l$ and m as subdegrees (The case (8) of Lemma 2 may be eliminated easily). The remaining cases are as follows.

(i) $\begin{cases} l = k(k-1) \\ m = k(k-1) \end{cases}$ and Δ, Γ, Λ are self-paired.

(ii) $\begin{cases} l = k(k-1) \\ m = k(k-1)/2 \end{cases}$

(iii) $\begin{cases} l = k(k-1)/2 \\ m = k(k-1) \end{cases}$

(iv) $\begin{cases} l = k(k-1)/2 \\ m = k(k-1)/2 \end{cases}$ and Δ, Γ, Λ are self-paired.

(v) $\begin{cases} l = k(k-1)/2 \\ m = k \end{cases}$ and Δ, Λ are paired orbits.

To eliminate these cases, the condition (*) above Step 1 are available. In the case (i) through the case (iv), parameters μ_1 and ν_1 involved in the coefficients of $g(x)$ (see Lemma 2.1 in [6]) are not determined by only the values of l and m. However, the structures of L and M, in particular, the structure of L give the values of μ_1 and ν_1 and we may eliminate these cases as in [6]. In the case (v) the structure of L or the decomposition of $1_{H|H}^{G} = 1_H + 1_{A_{k-1}}^{A_k} + 1_L^{A_k} + 1_M^{A_k}$ into irreducible characters as in [11] and [5] yield a contradiction.

For $k \leqq 15$, l and m are determined, and we examine each case as in [6]. On the occasion, Hall [2] and the table of the primitive groups of degree not exceeding 20 in Sims [10] make calculations easy. The case $k=7, l=42, m=126$ may be eliminated by Parrot [9] and by the fact that the representation of M_{22} (the Mathieu group of degree 22) on A_7 is of rank 3 (14 in Lüneburg [7]). Finally only the case $k=7$, $l=21$, $m=35$ remains and in fact, $A_7 \cdot E_6$ in Proposition gives this last case. Mr. H. Enomoto kindly informed that $A_7 \cdot E_6$ is the unique primitive extension of rank 4 of A_7.

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Directly after this work was done, more generally, E. Bannai has determined the primitive extensions of rank 4 of 5-ply transitive permutation groups (cf. E. Bannai: Primitive extensions of rank 4 of multiply transitive permutation groups, I, II (to appear)).

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