163. *Regularity of Solutions of Hyperbolic Mixed Problems with Characteristic Boundary*

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§1. Introduction. At first we recall the following well-known property of a solution of a hyperbolic Cauchy problem which is L^2-well posed: If the initial value is in $H^r(R^n)$, then the solution is also in $H^r(R^n)$ for any time >0. We call this "The property of having finite r-norm is persistent".

The author proved in [2] that, for a mixed problem to a first order hyperbolic system, if this mixed problem is L^2-well posed and the boundary is not characteristic for the equation, then the property of having finite r-norm is persistent.

In this note we discuss whether the persistent property holds or not in the case where the boundary is characteristic for the equation. Let Ω be a sufficiently smooth domain in R^n, $M=\partial/\partial t-L(t,x;D_x)$ be a first order hyperbolic system whose coefficients are $N\times N$ matrices in $\mathcal{B}([0,T]\times\Omega)$ and $P(t,x)$ be an $N\times N$ matrix defined on $[0,T]\times\partial\Omega$. Let us consider the mixed problem

$$(\mathrm{P})\begin{cases}(1.1) & M[u(t,x)]=f(t,x) & \text{in } [0,T]\times\Omega\\ (1.2) & u(0,x)=\varphi(x) & \text{on } \Omega\\ (1.3) & P(t,x)u(t,x)=0 & \text{on } [0,T]\times\partial\Omega.\end{cases}$$

Definition. The mixed problem (P) is said to be L^2-well posed if for any initial data $\varphi(x)\in D_0=\{u(x)\in H^1(\Omega)\,;\,P(0,x)u|_{\partial\Omega}=0\}$ and any second member $f(t,x)\in\mathcal{E}_t^0(H^1(\Omega))\cap\mathcal{E}_t^1(L^2(\Omega))$[1)] there exists a unique solution $u(t,x)$ of (P) in $\mathcal{E}_t^1(L^2(\Omega))\cap\mathcal{E}_t^0(\mathcal{D}(L(t)))$ satisfying the following energy inequality

$$(1.4)\qquad \|u(t)\|\leqq c(T)\left(\|\varphi\|+\int_0^t\|f(s)\|\,ds\right),\qquad t\in[0,T],$$

where $c(T)$ is a positive constant which depends only on T.

We remark that $\mathcal{D}(L(t))$ is the closure of $D_t=\{u(x)\in H^1(\Omega)\,;\,P(t)u|_{\partial\Omega}=0\}$ by the norm $\|u\|_{L(t)}=\|u\|+\|L(t)u\|$. At first we state

Theorem 1. *In the case where* $\Omega=R_+^2=\{(x,y)\,;\,x>0, y\in R^1\}$, $L=\begin{bmatrix}-2 & 0\\ 0 & 0\end{bmatrix}\partial/\partial x+\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\partial/\partial y$ *and* $P=[1\quad 0]$, *the mixed problem* (P) *is L^2-well posed, but the property of having finite r-norm is not persistent. More precisely, if the initial value* $\varphi(x,y)\in H^m(R_+^2)$ *satisfies*

1) $\mathcal{E}_t^k(E)$ is the set of E-valued functions of t which are k-times continuously differentiable.

$P(L^k\varphi)|_{x=0}=0$ $(k=0,1,\cdots,m-1)$ *and* $f(t,x,y)\equiv 0$, *then the solution* $u(t,x,y)={}^t(u_1,u_2)$ *has the following properties*

(i) $u_1(t,x,y)\in\mathcal{E}_t^p(H^{[(m+1-p)/2]}(R_+^2))$ *and* $u_2(t,x,y)\in\mathcal{E}_t^p(H^{[(m-p)/2]}(R_+^2))$ *for any* $p=0,1,\cdots,m$,

(ii) *moreover, if we suppose more strictly that* $\varphi(0,y)\notin H^m(R^1)$, *then* $u_1(t,x,y)\notin H^{[(m+1)/2]+1}(R_+^2)$ *and* $u_2(t,x,y)\notin H^{[m/2]+1}(R_+^2)$ *for any* $t>0$.

The above results can be extended to the following form. Let us consider

$$L(t,x\,;D_x)=\sum_{i=1}^n A_i(t,x)\frac{\partial}{\partial x_i}+B(t,x) \tag{1.5}$$

where A_i $(i=1,\cdots,n)$ and B are $N\times N$ matrices, and assume that the boundary $\partial\Omega$ of Ω is compact and sufficiently smooth. For simplicity, we assume the following conditions

(C.1) A_i $(i=1,\cdots,n)$ are Hermitian matrices,

(C.2) the boundary matrix $A_B=\sum_{i=1}^n A_i(t,x)\nu_i(x)$ is singular, but its rank is constant on $\partial\Omega$ where $\vec{n}=(\nu_1,\nu_2,\cdots,\nu_n)$ is the exterior unit normal to $\partial\Omega$,

(C.3) $P(t,x)$ is an $N\times N$ matrix, rank $P=l=$constant and Ker $P(t)$ is maximally non-positive for $L(t)$ on $\partial\Omega$, i.e., we assume that

$$u\cdot\overline{A_Bu}\leqq 0,\quad u\in\operatorname{Ker}P,\quad t\geqq 0,\quad x\in\partial\Omega,$$

and that Ker P is not properly contained in any other subspace having this property.

Then we have

Theorem 2. *Assume that the data* $\varphi(x)\in H^m(\Omega)$ *and the second member* $f(t,x)\in\mathcal{E}_t^m(L^2)\cap\mathcal{E}_t^{m-1}(H^1)\cap\cdots\cap\mathcal{E}_t^0(H^m)$ *satisfy the compatibility conditions* (1.6) *of order* $(m-1)$:

$$\sum_{i=1}^k\binom{k}{i}\frac{\partial^i P}{\partial t^i}(0,x)\cdot\varphi^{(k-i)}(x)\Big|_{\partial\Omega}=0,\qquad k=0,1,\cdots,m-1, \tag{1.6}$$

where $\varphi^{(0)}(x)=\varphi(x)$ *and* $\varphi^{(p+1)}(x)$ $(p\geqq 0)$ *is defined successively by the formula*

$$\begin{aligned}\varphi^{(p+1)}(x)=\sum_{i=1}^p\binom{p}{i}&\left(\sum_{j=1}^n\frac{\partial^i A_j}{\partial t^i}(0,x)\frac{\partial}{\partial x_j}+\frac{\partial^i B}{\partial t^i}(0,x)\right)\\&\times\varphi^{(p-i)}(x)+\frac{\partial^p f}{\partial t^p}(0,x).\end{aligned} \tag{1.7}$$

Then there exists a unique solution $u(t,x)$ *of* (P) *in* $\mathcal{E}_t^p(H^{[(m-p)/2]}(\Omega))$ $(p=0,1,\cdots,m)$, *and it does not necessarily belong to* $H^{[m/2]+1}(\Omega)$ *for any* $t>0$.

Remark. In Maxwell equation, we pose $\vec{n}\times\vec{E}\,|_{\partial\Omega}=0$ as the boundary condition where $\vec{n}$ is the exterior normal of the boundary and $\vec{E}$ is the electric field vector. Then this mixed problem satisfies the conditions (C.1), (C.2) and (C.3). However in this case the property of having finite r-norm is persistent.

Theorem 3. *In the case where* $\Omega=R^2_+$, $L=\begin{bmatrix}-a & 0\\ 0 & 0\end{bmatrix}\partial/\partial x+\begin{bmatrix}a_1 & \bar{c}\\ c & a_2\end{bmatrix}\times\partial/\partial y$ $(a>0;\, a_1, a_2\in R)$ *and* $P=[1\ 0]$, *the necessary and sufficient condition in order that the property of having finite r-norm be persistent is* $c=0$.

§2. Proof of Theorem 1. Since the L^2-well posedness is obvious in view of [1], we prove the latter of this theorem. If $\varphi(x,y)={}^t(\varphi_1,\varphi_2)$ is in $H^1(R^2_+)$ and $\varphi_1(0,y)=0$, then the solution $u(t,x,y)$ is given by $u(t)=e^{Lt}\varphi$, which is in $\mathcal{E}^1_t(L^2)\cap\mathcal{E}^0_t(\mathcal{D}(L))$. Since the coefficients of L is constant, $\partial u/\partial y(t,x,y)=e^{Lt}(\partial\varphi/\partial y)\in\mathcal{E}^0_t(L^2)$. From the equation it follows

$$\frac{\partial u_1}{\partial x}=\frac{1}{2}\left\{\frac{\partial u_2}{\partial y}-\frac{\partial u_1}{\partial t}\right\}\in\mathcal{E}^0_t(L^2).$$

Hence $u_1(t,x,y)$ is in $\mathcal{E}^0_t(H^1(R^2_+))$. Our purpose is to show that $u_2(t,x,y)$ does not belong to $H^1(R^2_+)$ for any $t>0$. Let us prove this by contradiction. For this we construct the solution $u(t,x,y)$ concretely by using Fourier-Laplace transform. We extend the definition domain of u to $R^1_+\times R^2$ by $u(t,x,y)=0$ for $x<0$, and denote by $\tilde{u}(t,\xi,\eta)$ the image of Fourier transform of $u(t,x,y)$, i.e.,

$$\tilde{u}(t,\xi,\eta)=\int_{R^2}e^{-i(x\xi+y\eta)}u(t,x,y)dxdy. \tag{2.1}$$

Then it follows

$$\begin{aligned}\tilde{u}_1(t,\xi,\eta)&=\frac{i\eta}{\alpha}\sin\alpha t\cdot e^{-i\xi t}\tilde{\varphi}_2(\xi,\eta)\\&\quad+\left(\cos\alpha t-\frac{i\xi}{\alpha}\sin\alpha t\right)e^{-i\xi t}\tilde{\varphi}_1(\xi,\eta),\end{aligned}\tag{2.2}$$

$$\begin{aligned}\tilde{u}_2(t,\xi,\eta)&=\frac{i\eta}{\alpha}\sin\alpha t\cdot e^{-i\xi t}\tilde{\varphi}_1(\xi,\eta)\\&\quad+\left\{\left(\cos\alpha t-\frac{i\xi}{\alpha}\sin\alpha t\right)+\frac{2i\xi}{\alpha}\sin\alpha t\right\}e^{-i\xi t}\tilde{\varphi}_2(\xi,\eta)\end{aligned}\tag{2.3}$$

where $\alpha=\sqrt{\xi^2+\eta^2}$. Since

$$\begin{aligned}\frac{\partial u_2}{\partial x}(t,x,y)&=\int_0^t\frac{\partial^2 u_2}{\partial s\partial x}(s,x,y)ds+\frac{\partial\varphi_2}{\partial x}(x,y)\\&=\frac{1}{2}\int_0^t\left\{\frac{\partial^2 u_2}{\partial y^2}(s,x,y)-\frac{\partial^2 u_1}{\partial s\partial y}(s,x,y)\right\}ds+\frac{\partial\varphi_2}{\partial x}(x,y)\\&=\frac{1}{2}\int_0^t\frac{\partial^2 u_2}{\partial y^2}(s,x,y)ds-\frac{1}{2}\left\{\frac{\partial u_1}{\partial y}(t,x,y)-\frac{\partial\varphi_1}{\partial y}(x,y)\right\}+\frac{\partial\varphi_2}{\partial x}(x,y),\end{aligned}$$

the necessary and sufficient condition in order that $u_2(t,x,y)$ be in $H^1(R^2_+)$ is

$$\int_0^t\frac{\partial^2 u_2}{\partial y^2}(s,x,y)ds\in L^2(R^2_+),\quad\text{i.e.,}\quad\int_0^t(i\eta)^2\tilde{u}_2(s,\xi,\eta)ds\in L^2(R^2). \tag{2.4}$$

Substituting (2.3) into (2.4), we get

$$
(2.5)\quad \begin{aligned}\int_0^t (i\eta)^2\tilde{u}_2(s,\xi,\eta)ds &= \frac{(i\eta)^3}{a}\cdot\tilde{\varphi}_1(\xi,\eta)\cdot\int_0^t \sin as\cdot e^{-i\xi s}ds\\ &\quad +(i\eta)^2\cdot\tilde{\varphi}_2(\xi,\eta)\cdot\int_0^t\left(\cos as-\frac{i\xi}{a}\sin as\right)e^{-i\xi s}ds\\ &\quad +\frac{(2i\xi)(i\eta)^2}{a}\tilde{\varphi}_2(\xi,\eta)\cdot\int_0^t e^{-i\xi s}\sin as\,ds\\ &= I_1+I_2+I_3.\end{aligned}
$$

The terms I_1 and I_2 are easily proved to be in $L^2(R^2)$, therefore the term I_3 must be in $L^2(R^2)$. Since

$$
\int_0^t e^{-i\xi s}\sin as\,ds=\frac{1}{a+|\xi|}(1-\cos at\cdot e^{-i\xi t})-\frac{i\xi}{a+|\xi|}\int_0^t e^{(\mathrm{sign}\,\xi\cdot a-\xi)si}ds,
$$

we get the following (2.6) in order to be $I_3\in L^2(R^2)$

$$
(2.6)\qquad \frac{(i\xi)^2(i\eta)^2}{a(a+|\xi|)}\tilde{\varphi}_2(\xi,\eta)\cdot\int_0^t e^{(\mathrm{sign}\,\xi\cdot a-\xi)si}ds\in L^2(R^2).
$$

Taking account of the identity

$$
i\xi\cdot\tilde{\varphi}_2(\xi,\eta)=\widetilde{\frac{\partial\varphi_2}{\partial x}}(\xi,\eta)+\tilde{\varphi}_2(0,\eta),\qquad \tilde{\varphi}_2(0,\eta)=\int_{R^1}e^{-iy\eta}\varphi_2(0,y)dy,
$$

we see that (2.6) is equivalent to

$$
(2.7)\qquad \tilde{F}(t,\xi,\eta)=\frac{(i\xi)(i\eta)^2}{a(a+|\xi|)}\tilde{\varphi}_2(0,\eta)\cdot\int_0^t e^{(\mathrm{sign}\,\xi\cdot a-\xi)si}ds\in L^2(R^2).
$$

If we put $l(t,\eta)=2(\eta^2t^2-\pi^2/16)/\pi t$, then for $|\xi|\geqq l(t,\eta)$

$$
(2.8)\qquad \left|\int_0^t e^{(\mathrm{sign}\,\xi\cdot a-\xi)si}ds\right|\geqq\frac{t}{\sqrt{2}}.
$$

Hence it follows

$$
\int_{R^1}\frac{\xi^2}{a^2(a+|\xi|)^2}\left|\int_0^t e^{(\mathrm{sign}\,\xi\cdot a-\xi)si}ds\right|^2d\xi\geqq\frac{ct^2}{\sqrt{l^2+t^2}}
$$

where $c=1/16\int_0^\infty t^2(1+t^2)^{-2}dt$. Therefore if we take $\varphi_2(x,y)$ as $\varphi_2(0,y)\notin H^1(R^1)$, then $\tilde{F}(t,\xi,\eta)$ does not belong to $L^2(R^2)$ for any $t>0$. This is a contradiction. Thus Theorem 1 is proved in the case $m=1$. For general m we can prove by induction.

§ 3. Proof of Theorem 2. We can prove as in [2] that this mixed problem has a finite propagation speed. Thus by the local transformation we can reduce to the case

$$
\Omega=R^n_+=\{(x_1,\cdots,x_n);(x_1,\cdots,x_{n-1})\in R^{n-1},x_n>0\}.
$$

Moreover, applying an appropriate transformation of unknown functions, we have only to consider the following fairly simple mixed problem

$$
(\mathrm{M})\begin{cases}(3.1)\quad \partial u/\partial t=(\sum_{i=1}^n A_i(t,x)\partial/\partial x_i+B(t,x))u+f(t,x)=L(t)u+f(t),\\ \qquad\qquad t>0,\quad x\in R^n_+,\\ (3.2)\quad u(0,x)=\varphi(x),\quad x\in R^n_+,\\ (3.3)\quad Pu|_{x_n=0}=0,\quad t>0,\quad (x_1,\cdots,x_{n-1})\in R^{n-1},\end{cases}
$$

where (A.1) A_i $(i=1,\cdots,n)$ are $N\times N$ Hermitian matrices and A_n

$=\begin{bmatrix}\tilde{A}_n & 0 \\ 0 & 0\end{bmatrix}$ where $\tilde{A}_n$ is an $r\times r$ non-singular matrix, (A.2) $P=[E_l O]$ is an $l\times N$ matrix where E_l is an $l\times l$ unit matrix, (A.3) $\operatorname{Ker} P$ is maximally non-positive for $L(t)$. We remark that (A.3) assures $l\leqq r$. Here we treat the mixed problem (M) when $L(t)$ is independent of t. When $L(t)$ depends on t, we can prove Theorem 2 by using energy inequalities and Cauchy's polygonal line as in [2]. Using Theorem 3.2 of Lax-Phillips [1], we see that L generates a semi-group $T(t)$ in $L^2(R^n_+)$, from which the L^2-well posedness is proved. We pass to the problem of regularity. Let us put $v_1={}^t(u_1,\cdots,u_r)$, $v_2={}^t(u_{r+1},\cdots,u_N)$, $g_1={}^t(f_1,\cdots,f_r)$ and $g_2={}^t(f_{r+1},\cdots,f_N)$, then there exist first order differential operators L_{ij} $(i,j=1,2)$ such that

$$(3.4)_i \qquad \partial v_i/\partial t=L_{i1}v_1+L_{i2}v_2+g_i, \qquad i=1,2.$$

We see from (A.1) that L_{12}, L_{21} and L_{22} don't contain the derivative with respect to x_n. First we consider the case $m=1$. Then the solution $u(t,x)$ is given by

$$u(t,x)=T(t)\varphi+\int_0^t T(t-s)f(s)ds$$

which is in $\mathcal{E}^1_t(L^2)\cap\mathcal{E}^0_t(\mathcal{D}(L))$. Let us put $U(t,x)={}^t({}^tu, {}^t\partial u/\partial t, {}^t\partial u/\partial x_1, \cdots, {}^t\partial u/\partial x_{n-1})$, then $U(t,x)$ satisfies

$$(3.5) \qquad \begin{cases}\partial U/\partial t=\tilde{L}U+F(t,x)\\ U(0,x)=\Phi(x)\\ \tilde{P}U|_{x_n=0}=0\end{cases}$$

where

$$\tilde{L}=\begin{bmatrix}L & & \\ & \ddots & \\ & & L\end{bmatrix}+\text{lower order}, \qquad \tilde{P}=\begin{bmatrix}P & & \\ & \ddots & \\ & & P\end{bmatrix}$$

$$\Phi(x)={}^t\left({}^t\varphi, {}^t(L\varphi+f(0)), \frac{{}^t\partial\varphi}{\partial x_1}, \cdots, \frac{{}^t\partial\varphi}{\partial x_{n-1}}\right)$$

and

$$F(t,x)={}^t\left({}^tf, \frac{{}^t\partial f}{\partial t}, {}^t\left(\frac{\partial f}{\partial x_1}-\frac{\partial A_n}{\partial x_1}\begin{bmatrix}\tilde{A}_n^{-1} & 0\\ 0 & 0\end{bmatrix}f\right), \cdots, {}^t\left(\frac{\partial f}{\partial x_{n-1}}-\frac{\partial A_n}{\partial x_{n-1}}\begin{bmatrix}\tilde{A}_n^{-1} & 0\\ 0 & 0\end{bmatrix}f\right)\right).$$

As $\Phi(x)\in L^2(R^n_+)$ and $F(t,x)\in\mathcal{E}^0_t(L^2)$, we see from (3.5) that $U(t,x)$ is in $\mathcal{E}^0_t(L^2)$. Therefore it follows from $(3.4)_1$ that $\partial v_1/\partial x_1$ is in $\mathcal{E}^0_t(L^2)$. Hence $v_1(t,x)$ is in $\mathcal{E}^0_t(H^1)$.

Next we pass to the case $m=2$. Since in (3.5) $\Phi(x)\in H^1(R^n_+)$ and $F(t,x)\in\mathcal{E}^0_t(H^1)\cap\mathcal{E}^1_t(L^2)$, we can apply the result obtained now to (3.5). Therefore, if we put $V_i={}^t({}^tv_i, {}^t\partial v_i/\partial t, {}^t\partial v_i/\partial x_1, \cdots, {}^t\partial v_i/\partial x_{n-1})$ $(i=1,2)$, $V_1(t,x)$ is in $\mathcal{E}^1_t(L^2)\cap\mathcal{E}^0_t(H^1)$. Hence in (3.5) $L_{21}v_1+g_2$ is in $\mathcal{E}^1_t(L^2)\cap\mathcal{E}^0_t(H^1)$. Since v_2 is free of boundary condition and L_{22} generates a semi-group in $L^2(R^n_+)$, $v_2(t,x)$ is also in $\mathcal{E}^1_t(L^2)\cap\mathcal{E}^0_t(H^1)$, which implies the required result in the case $m=2$. For general m we can prove this theorem by induction. The method used here is essentially the same as in [2].

The detailed proof will be given in a forthcoming paper.

References

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