22. Semi-linear Poisson's Equations

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§1. Semi-linear Poisson's equations. Let S be a separable, locally compact, non-compact Hausdorff space, and $C_0(S)$ be the completion with respect to the maximum norm of the space of real-valued continuous functions with compact supports defined on S. $C_0(S)$ is thus a Banach lattice.[1] Assume that we are given a *"non-negative" contraction semi-group* $\{T_t\}_{t\geqslant 0}$ *of class* (C_0) *in* $C_0(S)$ (see Phillips [11], Hasegawa [5] and Sato [12]). We shall be concerned with the situation in which

(1) the infinitesimal generator A of $\{T_t\}_{t\geqslant 0}$ admits a densely defined inverse A^{-1}.

That is, we suppose that the *semi-group* $\{T_t\}_{t\geqslant 0}$ *admits a "potential operator" V in the sense of Yosida* [17] (see also Chapter XIII, 9 of Yosida [19]):

$$V=-A^{-1}.$$

Now we introduce a nonlinear operator[2] β_0 in $C_0(S)$ associated with a strictly monotone increasing continuous function $\beta\colon D(\beta)=(a,b)\to R^1$, $-\infty\leq a<0<b\leq+\infty$, such that $\beta(0)=0$, $\lim_{r\downarrow a}\beta(r)=-\infty$ if $a\neq-\infty$, and that $\lim_{r\uparrow b}\beta(r)=+\infty$ if $b\neq+\infty$:

$$(2)\qquad \begin{aligned} &D(\beta_0)=\{u\in C_0(S)\,;\,u(s)\in D(\beta) && \text{for any } s\in S\},\\ &(\beta_0 u)(s)=\beta(u(s)),\ s\in S, && \text{for } u\in D(\beta_0). \end{aligned}$$

We consider the "semi-linear Poisson's equation":

$$Au-\beta_0 u=-f,\qquad f\in C_0(S).$$

Our theorem of the existence and uniqueness reads:

Theorem. *The operator $A-\beta_0$ admits a densely defined inverse* $(A-\beta_0)^{-1}$.

Remark. It is shown in Yosida [18] that the semi-group in $C_0(R^N)$ associated with the N-dimensional Brownian motion admits a potential operator in his sense even in the *recurrent* cases, i.e., $N=1$ or 2 (see also Sato [13] and Hirsch [6], where one finds studies on the existence of potential operators associated with spatially homogeneous Markov

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1) We shall make use of the notation in Banach lattice. See, e.g., Chapter XII, 3 of Yosida [19].

2) Throughout the paper the mappings are all single-valued.

processes on R^N). Thus our result can be applied to the concrete *semi-linear Poisson's equation*:

$$\frac{1}{2}\Delta u-\beta(u)=-f$$

in each R^N, $N\geqslant 1$.[3]

§ 2. Proof of Theorem. We begin with the following.

Lemma. (i) *The operator $A-\beta_0$ is "dissipative (s)" in $C_0(S)$*:

$$\tau(u-v,(A-\beta_0)u-(A-\beta_0)v)\leqslant 0 \tag{3}$$
$$\text{for all } u, v\in D(A)\cap D(\beta_0) \text{ and } \lambda>0;$$

where, by definition,

$$\tau(f,g)=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}(\|f+\varepsilon g\|-\|f\|), \qquad f,g\in C_0(S).$$

In particular, $A-\beta_0$ is "dissipative":

$$\|(\lambda u-Au+\beta_0 u)-(\lambda v-Av+\beta_0 v)\|\geqslant\lambda\|u-v\|$$

for all $u,v\in D(A)\cap D(\beta_0)$ and $\lambda>0$.

(ii) *The operator $A-\beta_0$ is "dispersive (s)" in $C_0(S)$*:

$$\sigma((u-v)^+,(A-\beta_0)u-(A-\beta_0)v)\leqslant 0 \tag{4}$$
$$\text{for all } u, v\in D(A)\cap D(\beta_0) \text{ and } \lambda>0;$$

where, by definition,

$$\sigma(f,g)=\inf_{\substack{b\in[0,\infty)\\ k\in C_0(S), f\wedge|k|=0}}\tau(f,(g+k)\vee(-bf)), \qquad f\geqslant 0.$$

In particular, $A-\beta_0$ is "dispersive":

$$\|\{(\lambda u-Au+\beta_0 u)-(\lambda v-Av+\beta_0 v)\}^+\|\geqslant\lambda\|(u-v)^+\|$$

for all $u,v\in D(A)\cap D(\beta_0)$ and $\lambda>0$.

(iii) *Moreover we have the range condition*:

$$R(\lambda I-A+\beta_0)=C_0(S) \quad \text{whenever} \quad \lambda>0. \tag{5}$$

Thus, by (i) and (iii), $\lambda(\lambda I-A+\beta_0)^{-1}$ exists and a contraction on $C_0(S)$ for each $\lambda>0$. Besides, in view of (ii), each $\lambda(\lambda I-A+\beta_0)^{-1}$ is "order-preserving":

$$f\leqslant g \quad \text{implies} \quad \lambda(\lambda I-A+\beta_0)^{-1}f\leqslant\lambda(\lambda I-A+\beta_0)^{-1}g. \tag{6}$$

Proof of Lemma. It is known that the operator A is dissipative (s) (see Remark 3 of Hasegawa [5]) and dispersive (s) (Theorem 1 of Sato [12]):

$$\tau(u,Au)\leqslant 0 \quad \text{and} \quad \sigma(u^+,Au)\leqslant 0 \quad \text{for} \quad u\in D(A).$$

So is the nonlinear operator $-\beta_0$:

$$\tau(u-v,\ -\beta_0 u+\beta_0 v)\leqslant 0 \quad \text{and} \quad \sigma((u-v)^+,\ -\beta_0 u+\beta_0 v)\leqslant 0$$

for $u,v\in D(\beta_0)$, since, by 6.2 of Sato [12],

$$\tau(f,g)=\max_{\{s\in S;|f(s)|=\|f\|\}}(\operatorname{sgn} f(s))g(s) \qquad f\neq 0$$
$$=\|g\| \qquad f=0$$

and

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3) Note that our interest consists in the unboundedness of the domain considered. Cf. Brezis-Strauss [3] (the Laplacian in R^N *does not* satisfy the condition (III) in § 1 of [3]).

$$\sigma(f,g)=\max_{\{s\in S;\, f(s)=\|f\|\}} g(s) \qquad f\geqslant 0,\ f\neq 0$$
$$=0 \qquad f=0.$$

Because $\tau(f,\cdot)$ and $\sigma(f,\cdot)$, for a fixed f, are both subadditive, we have (3) and (4). It is easily seen that the dissipativity(s) and the dispersivity(s) imply dissipativity and dispersivity respectively (cf. Lemma 1 of Hasegawa [5] and Lemma 4.1 of Sato [12]). Finally we prove (iii). (One can proceed as in Konishi [9].) We have only to show (5) with $\lambda=1$ (see, e.g., Lemma 4 of Ôharu [10]). Fix an arbitrary $f\in C_0(S)$. We define an everywhere defined monotone non-decreasing continuous function $\beta^f: D(\beta^f)=R^1\to R^1$ by

$$\beta^f(r)=\begin{cases}\beta((I+\beta)^{-1}(\|f\|)) & \text{if } r>(I+\beta)^{-1}(\|f\|)\\ \beta(r) & \text{if } r\in D(\beta) \text{ and } |r+\beta(r)|\leqslant\|f\|\\ \beta((I+\beta)^{-1}(-\|f\|)) & \text{if } r<(I+\beta)^{-1}(-\|f\|).\end{cases}$$

Define the corresponding operator $(\beta^f)_0$ in $C_0(S)$ by (2) with $\beta=\beta^f$. Thus $-(\beta^f)_0$ is everywhere defined continuous dissipative operator in $C_0(S)$. Accordingly, by Theorem I of Webb [16] (see also Theorem 1 of Barbu [1]), $R(I-A+(\beta^f)_0)=C_0(S)$, i.e., there exists $u\in D(A)$ such that

$$u-Au+(\beta^f)_0u=f. \tag{7}$$

On the other hand,

$$\begin{aligned}&\|(u+(\beta^f)_0u-\|f\|)^+\|\\ &\quad=\sigma((u+(\beta^f)_0u-\|f\|)^+,\ u+(\beta^f)_0u-\|f\|)\\ &\quad=\sigma((u-(I+\beta^f)^{-1}(\|f\|))^+,\ Au+f-\|f\|)\leqslant 0\end{aligned}$$

and, similarly,

$$\|(u+(\beta^f)_0u+\|f\|)^-\|\leqslant 0.$$

Hence

$$|u(s)+\beta^f(u(s))|\leqslant\|f\| \qquad \text{for } s\in S.$$

Therefore $u\in D(A)\cap D(\beta_0)$ and (7) is written as

$$u-Au+\beta_0u=f.$$

Q.E.D.

The following is a nonlinear version of a part of the *abelian ergodic theorems* (see, e.g., Lemma 1 in Chapter VIII, 4 and also (2) in Chapter XIII, 9 of Yosida [19]).

Proposition. *Let $\mathcal{A}$ be a (nonlinear) dissipative operator in a real Banach space $\mathcal{X}$:*

$$\|(\lambda I-\mathcal{A}u)-(\lambda I-\mathcal{A}v)\|\geqslant\lambda\|u-v\|, \qquad \textit{for } u,v\in D(\mathcal{A}) \textit{ and } \lambda>0$$

with

$$R(\lambda I-\mathcal{A})=\mathcal{X} \qquad \textit{for } \lambda>0.$$

Then

$$\overline{R(-\mathcal{A})}=\left\{f\in\mathcal{X};\ \lim_{\lambda\downarrow 0}\lambda(\lambda I-\mathcal{A})^{-1}f=0\right\}. \tag{8}$$

Proof. We denote by $\mathcal{M}$ the right-hand side of (8). $\mathcal{M}$ is closed since $\lambda(\lambda I-\mathcal{A})^{-1}$ are contractions. Set $f\in R(-\mathcal{A})$. Note that

$$R(-\mathcal{A})=R(-\mathcal{A}(I-\mathcal{A})^{-1})=R(I-(I-\mathcal{A})^{-1}).$$

Hence there exists $g \in \mathcal{X}$ satisfying $f = g - (I - \mathcal{A})^{-1} g$. By using the "nonlinear resolvent equation" (cf., e.g., Lemma 5 of Ôharu [10] or Lemma 1.2 of Crandall-Liggett [4]), we get

$$\begin{aligned}
&\|\lambda(\lambda I - \mathcal{A})^{-1} f\| \\
&\quad \leqslant \lambda \|(\lambda I - \mathcal{A})^{-1} f - (I - \mathcal{A})^{-1} g\| + \lambda \|(I - \mathcal{A})^{-1} g\| \\
&\quad = \lambda \|(\lambda I - \mathcal{A})^{-1} f - (\lambda I - \mathcal{A})^{-1}(g + (\lambda - 1)(I - \mathcal{A})^{-1} g)\| \\
&\qquad + \lambda \|(I + \mathcal{A})^{-1} g\| \\
&\quad \leqslant \|f - g - (\lambda - 1)(I - \mathcal{A})^{-1} g\| + \lambda \|(I - \mathcal{A})^{-1} g\| \\
&\quad = 2\lambda \|(I - \mathcal{A})^{-1} g\|.
\end{aligned}$$

Thus $f \in \mathcal{M}$. Therefore $\overline{R(-\mathcal{A})} \subset \overline{\mathcal{M}} = \mathcal{M}$. Next we set $f \in \mathcal{M}$. Then

$$\begin{aligned}
f &= \lim_{\lambda \downarrow 0} (f - \lambda(\lambda I - \mathcal{A})^{-1} f) \\
&= \lim_{\lambda \downarrow 0} (-\mathcal{A}(\lambda I - \mathcal{A})^{-1} f) \in \overline{R(-\mathcal{A})}.
\end{aligned}$$

Q.E.D.

Proof of the Theorem. By Proposition, in order to prove $\overline{R(A - \beta_0)} = C_0(S)$ we have to show

$$(9) \qquad \lim_{\lambda \downarrow 0} \lambda(\lambda I - A + \beta_0)^{-1} f = 0 \qquad \text{for } f \in C_0(S).$$

Note that, for $\lambda > 0$ and $f \in C_0(S)$, we have by (6)

$$\lambda(\lambda I - A + \beta_0)^{-1} f \leqslant \lambda(\lambda I - A + \beta_0)^{-1} f^+ \leqslant \lambda(\lambda I - A)^{-1} f^+,$$

$$\lambda(\lambda I - A + \beta_0)^{-1} f \geqslant \lambda(\lambda I - A + \beta_0)^{-1} f^- \geqslant \lambda(\lambda I - A)^{-1} f^-$$

(one finds a similar inequality in Konishi [8]).
In particular, we have that

$$|\lambda(\lambda I - A + \beta_0)^{-1} f| \leqslant \lambda(\lambda I - A)^{-1} |f|, \ \lambda > 0, \ f \in C_0(S),$$

and, therefore, that

$$\|\lambda(\lambda I - A + \beta_0)^{-1} f\| \leqslant \|\lambda(\lambda I - A)^{-1} |f|\|, \ \lambda > 0, \ f \in C_0(S).$$

Note that the condition (1) is equivalent to:

$$\lim_{\lambda \downarrow 0} \lambda(\lambda I - A)^{-1} f = 0 \qquad \text{for } f \in C_0(S)$$

(cf. Proposition 1 in Chapter XIII, 9 of Yosida [19]). Thus we have (9). Next we prove that $A - \beta_0$ is an injection:
Suppose that

$$Au - \beta_0 u = Av - \beta_0 v$$

for some pair $u, v \in D(A) \cap D(\beta_0)$. Then

$$\tau(u - v, \beta_0 u - \beta_0 v) = \tau(u - v, Au - Av) \leqslant 0,$$

from which follows that $u = v$. Q.E.D.

Comment. Our Theorem *might be* expressed also in the following form.

The semi-group $\{\exp(t(A - \beta_0))\}_{t \geqslant 0}$ *admits a "nonlinear potential operator"* V_β:

$$V_\beta = (-A + \beta_0)^{-1};$$

where $\{\exp(t(A - \beta_0))\}_{t \geqslant 0}$ is the nonlinear order-preserving semi-group of contractions on $\overline{D(A) \cap D(\beta_0)} \subset C_0(S)$, generated in the sense of Theorem I of Crandall-Liggett [4] (see also Theorem B of Konishi [7]):

$$\exp(t(A-\beta_0))\cdot f=\lim_{n\to\infty}\left(I-\frac{t}{n}A+\frac{t}{n}\beta_0\right)^{-n}f$$
$$=\lim_{n\to\infty}(T_{t/n}e^{-(t/n)\beta_0})^n f$$

$t\geqslant 0$, $f\in\overline{D(A)\cap D(\beta_0)}$. For the latter formula (*the Lie-Trotter product formula*), see, e.g., Theorem 3.2 of Brezis-Pazy [2]. Cf. the proof of Proposition (3.22) due to Brezis in Webb [16]. One can prove also that

$$\exp(t(A-\beta_0))\cdot f=\lim_{n\to\infty}\left(\left(I-\frac{t}{n}A\right)^{-1}\left(I+\frac{t}{n}\beta_0\right)^{-1}\right)^n f,$$

$t\geqslant 0$, $f\in\overline{D(A)\cap D(\beta_0)}$.

Further study. We can apply our techniques to obtain a result similar to our Theorem in the framework of Hilbert space L^2. In this case β need not be *strictly* monotone increasing. The study of this direction is stimulated by the recent works of Yosida [20] and Sato [14]. We can make corresponding study also in $L^p(1<p<\infty)$ but *not* in L^1; Note that the semi-group in $L^p(R^N)$ $(1<p<\infty)$ associated with the N-dimensional Brownian motion admits a potential operator in the sense of Yosida but the corresponding semi-group in $L^1(R^N)$ *does not* (see Theorem 1.5 of Watanabe [15]).[4] See also the author's paper: Note on potential operators on L^p (in preparation).

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