54. A Remark on the Flow near a Compact Invariant Set

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Introduction. The qualitative behavior of the flow near a compact invariant set was studied by several authors. For planer dynamical systems, I. Bendixson gave a solution to the problem of possible qualitative behavior of the flow near a rest point. For a dynamical system on a locally compact space, T. Ura and I. Kimura gave a description of the flow near a compact invariant set ([7]). Further discussions of the flow near a compact invariant set and/or a compact minimal set were done by T. Ura, T. Saito, N. P. Bhatia and D. Desbrow, always assuming the phase space is locally compact ([1]–[3], [5], [6], [8]). In this paper we shall give an example showing that the description as in [7] is not valid if we replace the assumption "locally compact" by the assumption "complete metric" on the phase space. This example also shows that the same is true for the discussions mentioned above.

1. Notations and basic theorems.

Let π be a dynamical system on a topological space X. $C(x), C^+(x)$ and $C^-(x)$ denote the orbit, the positive semiorbit and the negative semiorbit, respectively, through a point $x \in X$. The positive (negative) limit set of $x \in X$ is denoted by $L^+(x)$ $(L^-(x))$. $\phi \neq M \subset X$ is called an invariant set if $C(x) \subset M$ for all $x \in M$. $M \subset X$ is called a minimal set if M is a closed invariant set and does not contain any closed invariant proper subset. Let $M \subset X$ be a closed invariant set (a minimal set). M is said to be isolated from closed invariant sets (minimal sets) if there exists a neighborhood U of M such that U does not contain any closed invariant set (minimal set) except those contained in M. A compact invariant set is said to be positively (negatively) asymptotically stable if for each neighborhood U of M there exists a neighborhood V of M such that $C^+(x)$ $(C^-(x)) \subset U$ for all $x \in V$, and if $\{y \in X; L^+(y) \subset M\}$ $(\{y \in X; L^-(y) \subset M\})$ is a neighborhood of M.

Let X be locally compact, and $M \subset X$ a non-open compact invariant set isolated from closed invariant sets. The following theorem is due to T. Ura and I. Kimura ([17]).

Theorem. *One and only one of the following alternatives holds.*

(1) *M is positively asymptotically stable.*

(2) *M is negatively asymptotically stable.*

(3) *There exist points $x \notin M$ and $y \notin M$ such that $\phi \neq L^+(x) \subset M$*

and $\phi \neq L^-(y) \subset M$.

Corollary. *There does not exist any neighborhood U of M satisfying the following* $(*)$.

$$(*) \qquad L^+(x)=L^-(x)=\phi \qquad \text{for all } x \in U-M.$$

Proof. This is a direct consequence of Theorem.

2. An example of a flow on a complete metric space.

We denote by R the set of real numbers with the usual topology. Let C be the space of real valued continuous functions defined on R with compact open topology, i.e.,

$$C=\{f: R\to R \text{ continuous}\}$$

and for $f \in C$

$$\{V_\varepsilon^{[a,b]}(f)\}_{\varepsilon>0,\,-\infty<a<b<\infty}$$

is a fundamental system of neighborhoods of f, where

$$V_\varepsilon^{[a,b]}(f)=\{g \in C\,;\,|f(t)-g(t)|<\varepsilon \text{ for } t \in [a,b]\}.$$

As is well known, C is a complete metric space for this topology. Let $\pi: C\times R\to C$ be the dynamical system defined by the shift transformation ([4]), i.e.,

$$\pi(f,\tau)(t)=f^\tau(t)=f(t+\tau) \qquad \text{for } (f,\tau) \in C\times R,\ t \in R.$$

We shall define a subset $X_0 \subset C$ as follows.

$$X_0=\{g_u\}_{u\geqq 0},$$

where

$$g_0(t)=0 \qquad \text{for all } t \in R,$$

and for $u>0$

$$g_u(t)=\begin{cases} \dfrac{1}{u}\left|t+\dfrac{1}{u}\right|+u & \text{for } t\leqq -\dfrac{1}{u} \\ u & \text{for } |t|<\dfrac{1}{u} \\ \dfrac{1}{u}\left|t-\dfrac{1}{u}\right|+u & \text{for } t\geqq \dfrac{1}{u}. \end{cases}$$

Let X be the invariant set with respect to π generated by X_0, i.e.,

$$X=\{g_u^\tau\}_{u\geqq 0,\tau \in R},$$

where $g_u^0=g_u$ for all $u\geqq 0$. The following Lemma 1 is an easy consequence of the definition of X_0.

Lemma 1. *If $u_n\to u_0$ $(u_n, u_0\geqq 0)$, then $g_{u_n}\to g_{u_0}$ in C.*

Lemma 2. *If $u_n\to u_0$ and $\tau_n\to\tau_0$, then $g_{u_n}^{\tau_n}\to g_{u_0}^{\tau_0}$ in C.*

Proof. Since by Lemma 1 $g_{u_n}^{\tau_n}=\pi(g_{u_n},\tau_n)$ and $g_{u_n}\to g_{u_0}$, we have

$$\pi(g_{u_n},\tau_n)\to\pi(g_{u_0},\tau_0)=g_{u_0}^{\tau_0}$$

by continuity of π, whence

$$g_{u_n}^{\tau_n}\to g_{u_0}^{\tau_0} \qquad \text{in } C.$$

Lemma 3. *Let $0<u<a<b<\infty$, and $-\infty<\alpha<\beta<\infty$. If $a\leqq g_u(t) \leqq b$ for all $t \in [\alpha,\beta]$, then we have*

$$\beta-\alpha\leqq u(b-a).$$

Proof. Since $g_u(t)=u<a$ for $|t|<1/u$, we have $[\alpha,\beta]\subset[1/u,\infty)$ or $[\alpha,\beta]\subset(-\infty,-1/u]$. Since $g_u(t)=g_u(-t)$ for all $t\in R$, we can assume, without loss of generality, that $[\alpha,\beta]\subset[1/u,\infty)$. Then $a\leqq g_u(\alpha)\leqq g_u(\beta)\leqq b$, whence

$$g_u(\beta)-g_u(\alpha)=\frac{1}{u}\left(\beta-\frac{1}{u}\right)+u-\frac{1}{u}\left(\alpha-\frac{1}{u}\right)-u$$
$$=\frac{1}{u}(\beta-\alpha)\leqq b-a.$$

Consequently we have

$$\beta-\alpha\leqq u(b-a).$$

Lemma 4. *Let* $0<u\leqq b<\infty$. *Then* $g_u(t)\leqq b$ *if and only if* $|t|\leqq u(b-u)+1/u$.

Proof. If $|t|\leqq 1/u$, then obviously $g_u(t)\leqq b$. If $t>1/u$, then

$$g_u(t)=\frac{1}{u}\left(t-\frac{1}{u}\right)+u\leqq b.$$

Hence we have

$$t\leqq u(b-u)+\frac{1}{u}.$$

Since $g_u(t)=g_u(-t)$ for all $t\in R$, the Lemma is proved.

Proposition 1. *The invariant set X is a closed subset of C, and so a complete metric space.*

Proof. Let $g_{u_n}^{\tau_n}\to g^*$ in C. We shall prove $g^*\in X$. If $g^*\equiv 0$, then obviously $g^*=g_0\in X$. Accordingly we can assume $g^*\not\equiv 0$. Thus there exists $t_0\in R$ such that $g^*(t_0)=\delta>0$. Since $g^*\in C$, there exist $\varepsilon>0$ and $T>0$ such that

$$\delta-\varepsilon<g^*(t)<\delta+\varepsilon \qquad \text{for } t\in[t_0-T,t_0+T].$$

Put

$$V=\{h\in C\,;\,h(t)\in(\delta-\varepsilon,\delta+\varepsilon) \text{ for } t\in[t_0-T,t_0+T]\}.$$

Then V is a neighborhood of g^*. Hence there exists an integer N such that $g_{u_n}^{\tau_n}\in V$ for all $n\geqq N$.

(1) We shall prove $\delta+\varepsilon\geqq u_n\geqq T/\varepsilon$ for $n\geqq N$.

In fact, since $g_{u_n}^{\tau_n}(t)\geqq u_n$ for all $t\in R$ and $g_{u_n}^{\tau_n}\in V$ for $n\geqq N$, we have $u_n\leqq\delta+\varepsilon$ for $n\geqq N$. Further, since

$$\delta-\varepsilon\leqq g_{u_n}^{\tau_n}(t)=g_{u_n}(t+\tau_n)\leqq\delta+\varepsilon \qquad \text{for } t\in[t_0-T,t_0+T],$$

we have by Lemma 3

$$2T\leqq 2\varepsilon u_n \qquad \text{for } n\geqq N.$$

Hence we have

$$u_n\geqq\frac{T}{\varepsilon} \qquad \text{for } n\geqq N,$$

i.e.,

$$\delta+\varepsilon\geqq u_n\geqq\frac{T}{\varepsilon}>0 \qquad \text{for } n\geqq N.$$

(2) We shall show that $\{\tau_n\}$ is bounded.

In fact, since $g_{u_n}^{\tau_n}(t_0)=g_{u_n}(\tau_n+t_0)\leqq\delta+\varepsilon$ for $n\geqq N$, we have by Lemma 4

$$\begin{aligned}|\tau_n+t_0|&\leqq u_n(\delta+\varepsilon-u_n)+\frac{1}{u_n}\\&\leqq(\delta+\varepsilon)\left(\delta+\varepsilon-\frac{T}{\varepsilon}\right)+\frac{\varepsilon}{T}\qquad\text{for } n\geqq N.\end{aligned}$$

Hence $\{\tau_n\}$ is bounded.

By (1) and (2) there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $u_{n(k)}\to u_0\geqq 0$ and $\tau_{n(k)}\to\tau_0$. By Lemma 2 we have

$$g_{u_{n(k)}}^{\tau_{n(k)}}\to g_{u_0}^{\tau_0}\qquad\text{as } k\to\infty.$$

Since $g_{u_n}^{\tau_n}\to g^*$, we have $g^*=g_{u_0}^{\tau_0}\in X$. Hence X is a closed subset of C. The second assertion is an obvious consequence of the first, since C is complete.

We denote the restriction of π to X by the same letter π. Put $M=\{g_0\}$. Then, since g_0 is a singular point of π, M is a compact minimal set (non-open), *a fortiori*, a compact invariant set.

Proposition 2. *The set M constructed above is isolated from closed invariant sets, and* (*) *in Corollary holds for every neighborhood V of M.*

Proof. Assume $V=V_\varepsilon^{[a,b]}(g_0)$ $(-\infty<a<b<\infty,\varepsilon>0)$ and $f=g_u^\tau\in V-M$. Since for the fixed $s\in R$

$$(**)\qquad \lim_{t\to{+\infty \atop -\infty}}\pi(g_u^\tau,t)(s)=\lim_{t\to{+\infty \atop -\infty}}g_u(\tau+t+s)={+\infty \atop -\infty},$$

there exists $t_0>0$ such that

$$\pi(g_u^\tau,t_0)(a)=g_u(\tau+t_0+a)>\varepsilon,$$

whence $g_u^{\tau+t_0}\notin V$. Consequently, V does not contain any closed invariant set except M, i.e., M is isolated from closed invariant sets. Further since (**) implies $L^+(f)=L^-(f)=\phi$, (*) in Corollary holds. Q.E.D.

Thus Corollary, and *a fortiori*, Theorem in § 1 are disproved for dynamical systems on a complete metric space. For the more detailed results obtained by those mentioned in the introduction, it is easy to see that each of them implies Corollary. So we see that none of these detailed results is true for dynamical systems on a complete metric space if we don't assume some additional property on the latter.

Remark. One easily verifies that our conclusion also holds for *local* dynamical systems, since the problem is local.

Remark. The space X in the example is arcwise connected and separable.

References

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