99. On Expandability

By Yûkiti KATUTA

(Comm. by Kenjiro SHODA, M. J. A., June 12, 1973)

In [1] Katětov proved the following useful theorem:

A normal space X is collectionwise normal and countably paracompact if and only if

(*) for every locally finite collection $\{F_\lambda \mid \lambda \in \Lambda\}$ of subsets of X there exists a locally finite collection $\{G_\lambda \mid \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset G_\lambda$ for every $\lambda \in \Lambda$.

Recently, Krajewski [3] has called a topological space X *expandable* if X satisfies this condition (*). Smith and Krajewski [4] have introduced some generalizations (almost expandability, etc.) of expandability, and they have obtained various results concerning these notions.

In this paper, we shall introduce new notions of θ-expandability, subexpandability etc., and obtain analogous results. Furthermore, we shall study additional properties of expandable spaces, θ-expandable spaces etc.

The proofs and details of the results will be published elsewhere.

1. A collection $\mathfrak{A}$ of subsets of a space X is said to be *bounded locally finite* [2], if there exists a positive integer n such that every point of X has a neighborhood which intersects at most n elements of $\mathfrak{A}$. Every discrete collection is bounded locally finite and every bounded locally finite collection is locally finite.

A space X is said to be *θ-expandable* (resp. *boundedly θ-expandable* or *discretely θ-expandable*), if for every locally finite (resp. bounded locally finite or discrete) collection $\{F_\lambda \mid \lambda \in \Lambda\}$ of subsets of X there exists a sequence $\mathfrak{G}_n = \{G_{\lambda,n} \mid \lambda \in \Lambda\}$, $n = 1, 2, \cdots$, of collections of open subsets of X satisfying the following two conditions:

(1) $F_\lambda \subset G_{\lambda,n}$ for each $\lambda \in \Lambda$ and each n.

(2) For each point x of X there exists a positive integer n such that only finitely many elements of $\mathfrak{G}_n$ contain x.

Theorem 1.1. (a) *X is boundedly θ-expandable if and only if X is discretely θ-expandable.*

(b) *X is θ-expandable if and only if X is discretely θ-expandable and countably θ-refinable.*

(c) *A θ-refinable space is θ-expandable.*

A space X is said to be *discretely subexpandable,* if for every discrete collection $\{F_\lambda \mid \lambda \in \Lambda\}$ of subsets of X there exists a sequence

$\mathfrak{G}_n=\{G_{\lambda,n} \mid \lambda \in \Lambda\}$, $n=1,2,\cdots$, of collections of open subsets of X satisfying the following two conditions:

(3) $F_\lambda \subset G_{\lambda,n}$ for each $\lambda \in \Lambda$ and for each n.

(4) For each point x of X there exists a positive integer n such that at most one element of $\mathfrak{G}_n$ contains x.

A discretely subexpandable space X is said to be *subexpandable* (resp. *boundedly subexpandable*), if it is countably subparacompact (resp. finitely subparacompact[1]).

Theorem 1.2. (a) *A subparacompact space is subexpandable.*

(b) *A space whose every closed subset is a G_δ-subset is subexpandable.*

(c) *A collectionwise normal space is boundedly subexpandable.*

2. An open covering of a space is said to be an *A-covering* [2], if it has a locally finite (not necessarily open) refinement. Every countable open covering is an A-covering. A covering $\mathfrak{U}$ is said to be *directed* [5], if for every pair (U, V) of elements of $\mathfrak{U}$ there exists an element W of $\mathfrak{U}$ such that $U \cup V \subset W$.

Theorem 2.1. *The following are equivalent for a space X:*

(a) *X is expandable.*

(b) *Every A-covering of X has a locally finite open refinement.*

(c) *Every directed A-covering of X has a locally finite open refinement.*

(d) *Every directed A-covering of X has a locally finite closed refinement.*

(e) *Every directed A-covering of X has an open locally star-refinement.*[2]

(f) *Every directed A-covering of X has an open cushioned refinement.*

Theorem 2.2. *The following are equivalent for a space X:*

(a) *X is almost expandable.*

(b) *Every A-covering of X has a point-finite open refinement.*

(c) *Every directed A-covering of X has a point-finite open refinement.*

(d) *Every directed A-covering of X has an open Δ-refinement.*

(e) *Every directed A-covering of X has a cushioned refinement.*

Theorem 2.3. *The following are equivalent for a space X:*

---

1) A space X is said to be *finitely subparacompact*, if every finite open covering of X has a σ-discrete closed refinement.

2) Let $\mathfrak{U}$ and $\mathfrak{V}$ be two coverings of a space X. If every point of X has a neighborhood W such that $St(W,\mathfrak{V}) \subset U$ for some $U \in \mathfrak{U}$, then we say that the covering $\mathfrak{V}$ is a *locally star-refinement* of the covering $\mathfrak{U}$. Obviously, every open star-refinement is a locally star-refinement and every locally star-refinement is a Δ-refinement.

(a) *X is θ-expandable.*

(b) *For every A-covering $\mathfrak{U}$ of X there is a sequence $\mathfrak{V}_n$, $n=1,2,\cdots$, of open refinements of $\mathfrak{U}$ such that for every point x of X there is some $\mathfrak{V}_n$ of which only finitely many elements contain x.*

(c) *For every directed A-covering $\mathfrak{U}$ of X there is a sequence $\mathfrak{V}_n$, $n=1,2,\cdots$, of open refinements of $\mathfrak{U}$ such that for every point x of X there is some $\mathfrak{V}_n$ of which only finitely many elements contain x.*

(d) *For every directed A-covering $\mathfrak{U}$ of X there is a sequence $\mathfrak{V}_n$, $n=1,2,\cdots$, of open refinements of $\mathfrak{U}$ such that for every point x of X there is some $\mathfrak{V}_n$ and some $U\in\mathfrak{U}$ with $St(x,\mathfrak{V}_n)\subset U$.*

(e) *Every directed A-covering of X has a σ-cushioned refinement.*

An open covering is said to be a *B-covering* [2], if it has a bounded locally finite refinement. Using B-coverings instead of A-coverings, we obtain analogous characterizations of bounded expandability, bounded almost expandability and bounded θ-expandability.

An open covering $\{U_\lambda \mid \lambda\in\Lambda\}$ is said to be a *C-covering*, if it satisfies that $U_\lambda=\bigcap_{\mu\neq\lambda}\bigcup_{\nu\neq\mu}U_\nu$ for every $\lambda\in\Lambda$. Every C-covering is a B-covering.

Theorem 2.4. *The following are equivalent for a space X:*

(a) *X is boundedly subexpandable.*

(b) *For every C-covering $\mathfrak{U}$ of X there is a sequence $\mathfrak{V}_n$, $n=1,2,\cdots$, open refinements of $\mathfrak{U}$ such that for every point x of X there is some $\mathfrak{V}_n$ of which only one element contains x.*

(c) *For every C-covering $\mathfrak{U}$ of X there is a sequence $\mathfrak{V}_n$, $n=1,2,\cdots$, open refinements of $\mathfrak{U}$ such that for every point x of X there is some $\mathfrak{V}_n$ and some $U\in\mathfrak{U}$ with $St(x,\mathfrak{V}_n)\subset U$.*

(d) *Every C-covering has a σ-discrete closed refinement.*

(e) *Every C-covering has a σ-locally finite closed refinement.*

(f) *Every C-covering has a σ-cushioned refinement.*

3. In [3] and [4], it has been shown that a θ-refinable space is paracompact (resp. metacompact), if and only if it is expandable (resp. almost expandable). Similarly we have

Theorem 3.1. *A θ-refinable space is subparacompact if and only if it is subexpandable.*

4. The following mapping theorem, sum theorem and subset theorem for θ-expandable spaces and subexpandable spaces hold.

Theorem 4.1. *Let $f: X\to Y$ be a continuous, closed onto mapping. If X is θ-expandable (resp. subexpandable), then Y is θ-expandable (resp. subexpandable)*

Theorem 4.2. *Let $\mathfrak{F}$ be a σ-locally finite closed covering of a space X. If every member of $\mathfrak{F}$ is θ-expandable (resp. subexpandable), then X is θ-expandable (resp. subexpandable).*

Theorem 4.3. *Every F_σ-subset of a θ-expandable (resp. subexpandable) space is θ-expandable (resp. subexpandable).*

For θ-expandable spaces, furthermore, the following mapping theorem and product theorem hold.

Theorem 4.4. *Let $f: X \to Y$ be a perfect mapping. If Y is θ-expandable, then X is θ-expandable.*

Theorem 4.5. *The product space of a θ-expandable space and a compact space is θ-expandable.*

5. A space X is said to be *countably expandable* if for every countable locally finite collection $\{F_n \mid n=1,2,\cdots\}$ of subsets of X there exists a countable locally finite collection $\{G_n \mid n=1,2,\cdots\}$ of open subsets of X such that $F_n \subset G_n$ for every n. Similarly, *countably θ-expandable*, etc. are defined. In [3], it has been shown that a space is countably expandable if and only if it is countably paracompact.

Theorem 5.1. *The following are equivalent for a space X:*

(a) *X is countably almost expandable.*

(b) *X is countably metacompact.*

(c) *X is countably θ-expandable.*

(d) *X is countably θ-refinable.*

Theorem 5.2. *A space is countably subexpandable if and only if it is countably subparacompact.*

Theorem 5.3. *The following are equivalent for a normal space X:*

(a) *X is countably expandable.*

(b) *X is countably almost expandable.*

(c) *X is countably θ-expandable.*

(d) *X is countably subexpandable.*

References

[1] M. Katětov: On extension of locally finite coverings. Colloq. Math., **6**, 145–151 (1958).

[2] Y. Katuta: On strongly normal spaces. Proc. Japan Acad., **45**, 692–695 (1969).

[3] L. L. Krajewski: On expanding locally finite collections. Canad. J. Math., **23**, 58–68 (1971).

[4] J. C. Smith and L. L. Krajewski: Expandability and collectionwise normality. Trans. Amer. Math. Soc., **160**, 437–451 (1971).

[5] J. Mack: Directed covers and paracompact spaces. Canad. J. Math., **19**, 649–654 (1967).