95. On Strongly Regular Rings

By Katsuo CHIBA and Hisao TOMINAGA
Department of Mathematics, Okayama University

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A ring R is called *strongly regular* if for every element $a \in R$ there exists an element $x \in R$ such that $a = a^2x$. As is well-known, R is strongly regular if and only if one of the following equivalent conditions is satisfied:

(A) For every element $a \in R$ there holds $a \in aR$ and there exists a central idempotent e such that $aR = eR$.

(B) R is a regular ring without nonzero nilpotent elements. Obviously, the notion "strongly regular" is right-left symmetric. Next, a ring R is called a *right* [*left*] *duo ring* if every right [left] ideal of R is an ideal. Finally, a ring R is called a *right* [*left*] *V-ring* if $R^2 = R$ and every right [left] ideal of R is an intersection of maximal right [left] ideals of R.

It is the purpose of this note to prove the following that contains [2; Theorem 2], [5; Theorem] and [7; Theorem 3 and Corollary 1]:

Theorem. *The following conditions are equivalent*:

(1) *R is strongly regular.*

(2) *R is a regular ring and is a subdirect sum of division rings.*

(3) *$\mathfrak{l} \cap \mathfrak{r} = \mathfrak{l}\mathfrak{r}$ for every left ideal $\mathfrak{l}$ and every right ideal $\mathfrak{r}$ of R.*

(4) *R contains no nonzero nilpotent elements and $R/\mathfrak{p}$ is regular for every prime ideal $\mathfrak{p} \subsetneqq R$.*

(5) *R is a regular, right duo ring.*

(6) *$\mathfrak{r} \cap \mathfrak{r}' = \mathfrak{r}\mathfrak{r}'$ for each right ideals $\mathfrak{r}, \mathfrak{r}'$ of R.*

(7) *R is a right duo ring such that every ideal is idempotent.*

(8) *R is a right duo, right V-ring.*

(9) *R contains no nonzero nilpotent elements and every completely prime ideal $\subsetneqq R$ is a maximal right ideal.*

(5′)–(9′). *The left-right analogues of* (5)–(9).

In the proof of our theorem, we shall use several familiar results, which are summarized in the next lemma.

Lemma. *Let R be a ring without nonzero nilpotent elements, and let a, b be elements of R.*

(a) *If $ab = 0$ then $ba = 0$, and so the right annihilator $r(a)$ coincides with the left one $l(a)$.*

(b) *If a is nonzero then $R/r(a)$ contains no nonzero nilpotent elements and the residue class $\bar{a}$ of a* mod *$r(a)$ is a non-zero-divisor.*

(c) *If R is a prime ring then R contains no nonzero zero-divisors.*

Proof of Theorem. (2)$\Rightarrow$(1)$\Rightarrow$(4), (1)$\Rightarrow$(5)$\Rightarrow$(6)$\Rightarrow$(7), (1) (and (6)) $\Rightarrow$(3)$\Rightarrow$(6): These are easily seen.

(7)$\Rightarrow$(1): Let a be an arbitrary element of R, and (a) the (right) ideal generated by a. Then, $(a)=(a)^2=(a)R(a)=aR(a)=(a^2)$, whence it follows that $a=a^2x$ with some x.

(1)$\Rightarrow$(2): Since a regular ring is semi-simple, it suffices to prove that a strongly regular prime ring R is a division ring. Given a nonzero $a\in R$, there exists a central idempotent e such that $a\in aR=eR$. Since $e(x-ex)=0$ for every $x\in R, R=eR=aR$ by Lemma (c). Hence, e is the identity of R and a is invertible.

(1)$\Rightarrow$(8): It remains only to prove that an arbitrary ideal $\mathfrak{a}$ of R is an intersection of maximal ideals. Let b be not in $\mathfrak{a}$, and e a central idempotent such that $b\in bR=eR$. There exists then an ideal $\mathfrak{m}\supseteq\mathfrak{a}$ which is maximal with respect to the exclusion of b. Since the set $\{e\}$ is multiplicatively closed and $\mathfrak{m}$ is maximal with respect to the exclusion of $\{e\}$, $\mathfrak{m}$ is a prime ideal. As was shown in the proof of (1)$\Rightarrow$(2), $R/\mathfrak{m}$ is a division ring, nemely, $\mathfrak{m}$ is maximal.

(8)$\Rightarrow$(1): Suppose that there exists an element a not contained in a^2R. We can find then a maximal (right) ideal $\mathfrak{m}$ such that $a^2R\subseteq\mathfrak{m}$ and $a\notin\mathfrak{m}$. Since $R/\mathfrak{m}$ is a division ring, we have $a^3\notin\mathfrak{m}$, which contradicts $a^2R\subseteq\mathfrak{m}$.

(4)$\Rightarrow$(9): This is obvious by the proof of (1)$\Rightarrow$(2).

(9)$\Rightarrow$(1): Let a be a nonzero element of R. Then, by Lemma (b), $\bar{R}=R/r(a)$ contains no nonzero nilpotent elements, $\bar{a}$ is a non-zero-divisor of $\bar{R}$, and every completely prime ideal $\subsetneq\bar{R}$ is a maximal right ideal of $\bar{R}$. Now, let M be the multiplicative semigroup generated by all the elements $\bar{a}-\bar{a}^2\bar{x}$ $(x\in R)$. Although the existence of the identity of $\bar{R}$ is not assumed, we may write $\bar{a}-\bar{a}^2\bar{x}=\bar{a}(1-\bar{a}\bar{x})$. First, we claim that M contains 0. In fact, if not, there exists a completely prime ideal $\bar{\mathfrak{p}}$ excluding M (see [1]). However, the existence of the inverse of $\bar{x}$ mod $\bar{\mathfrak{p}}$ yields a contradiction. Now, let $\bar{a}(1-\bar{a}\bar{x}_1)\cdots\bar{a}(1-\bar{a}\bar{x}_n)=0$, where n is chosen to be minimal. If $n>2$ then $(1-\bar{a}\bar{x}_1)\cdots\bar{a}(1-\bar{a}\bar{x}_n)=0$ yields a contradiction $\bar{a}\{(1-\bar{a}\bar{x}_n)(1-\bar{a}\bar{x}_1)\}\cdots\bar{a}(1-\bar{a}\bar{x}_{n-1})=0$ (Lemma (a)). Next, if $n=2$ then $(1-\bar{a}\bar{x}_1)\bar{a}(1-\bar{a}\bar{x}_2)\bar{a}^2=0$ yields $(1-\bar{a}\bar{x}_2)\bar{a}\bar{a}(1-\bar{a}\bar{x}_1)=0$, and hence $\bar{a}(1-\bar{a}\bar{x}_1)(1-\bar{a}\bar{x}_2)=0$ again by Lemma (a). We have seen therefore $a-a^2x_1\in r(a)=l(a)$, whence it follows $(a-a^2x_1)^2=0$, namely, $a=a^2x_1$.

Remark. In [7; Theorem 3], E. T. Wong proves also that if R is a strongly regular ring with 1 then for each $a\in R$ there exists a unit u such that $a^2u=a$. But, G. Ehrlich [3; Theorem 3] has proved the same with an elementary proof. Next, as a corollary to our theorem, we have the following theorem due to R. Hamsher: A commutative ring R is regular if and only if it has no nonzero nilpotent elements and every prime ideal $\subsetneqq R$ is maximal. Combining this with a theorem of W. Krull [4; Satz 10], we obtain at once the result of H. Lal [6; Theorem]: A commutative ring R with 1 is regular if and only if every primary ideal $\subsetneqq R$ is maximal.

References

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