94. *Codimension 1 Foliations on Simply Connected 5-Manifolds*

By Kazuhiko FUKUI
Mathematical Institute, Kyoto University

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1. Recently N. A'Campo [1] has shown that every simply connected, closed 5-manifold with vanishing second Stiefel-Whitney class admits a codimension 1 foliation. The essential point in his construction is to utilize Smale's classification theorem [4].

In this note, similarly utilizing Barden's result [2], we show that every simply connected, closed 5-manifold admits a codimension 1 foliation. All the manifolds and the foliations considered here, are smooth of class C^∞.

2. Preliminaries. a) The second Stiefel-Whitney class $\omega^2(M)$ of a simply connected manifold M may be regarded as a homomorphism $\omega^2: H_2(M:Z)\to Z_2$, and we may consider ω^2 to be non-zero on at most one element of a basis. In a simply connected 5-manifold, the value of ω^2 on the homology class carried by an imbedded 2-sphere is the obstruction to the triviality of its normal bundle. Such a "non-zero valued" class has order 2^i for some positive integer i. Then i is a diffeomorphism invariant $i(M)$ of M.

D. Barden [2] has classified simply connected, closed, smooth 5-manifolds under diffeomorphism. Such a manifold is determined by $H_2(\ )$ and $i(\ )$. More precisely:

Proposition 1 [2]. *Simply connected, closed, smooth, oriented 5-manifolds are classified under diffeomorphism as follows. A canonical set of representatives is* $\{X_j \# M_{k_1} \# \cdots \# M_{k_s}\}$, *where* $-1\leqq j\leqq\infty$, $s\geqq 0$, $1<k_1$ *and* k_i *divides* k_{i+1} *or* $k_{i+1}=\infty$. *A complete set of invariants is provided by* $H_2(M)$ *and* $i(M)$. (for the notation, see [2], p. 373.)

b) S^2-bundles over S^2 with group SO_3 are classified by $\pi_1(SO_3)\cong Z_2$. We denote by A the product, and by B the non-trivial bundle. Next consider reductions of the structure group to SO_2, which are classified by $\pi_1(SO_2)\cong Z$. Let T_k be the S^2-bundle associated with the reduction given by the integer k. Furthermore, let x be the class in $H_2(T_k)$ of the sphere imbedded as the cross-section, corresponding to the "south pole", and y be the class of the sphere imbedded as a fiber. If $\cdot$ denotes the intersection number of homology class, then $x\cdot x=k$, $x\cdot y=1$ (we have the orientation of y to ensure this) and $y\cdot y=0$. For the homology bases of A, B, we shall reduce the bundles as T_0, T_1. Then we have, in [5]

Proposition 2. *Let N be a simply connected 4-manifold, $\omega \in H_2(N)$ with $\omega\cdot\omega=2s$, then $N \# T_k$ admits a diffeomorphism inducing the following automorphism of $H_2(N\# T_k)$:*

$$\xi \in H_2(N)\to\xi-(\xi\cdot\omega)y,\ x\to x+\omega-sy,\ y\to y.$$

Generators x, y of the second homology groups of various copies of the 2-sphere bundles A, B will carry the same suffixes as the bundles. Now, consider for the case $N=A_1, T_k=A_2$. (i.e., k is even.) Put $\omega=ly_1$. By Proposition 2, we have a diffeomorphism $d_l: A_1\# A_2\to A_1\# A_2$ for each $1<l<\infty$. Let $e: A_1\# A_2\to A_1\# A_2$ be a diffeomorphism which induces the automorphism of H_2: $x_1, y_1, x_2, y_2\to y_2, x_2, y_1, x_1$. Put $\alpha(l)=d_l\cdot e$. Next consider for the case $N=B_1, T_k=B_2$. (i.e., k is odd.) Put $\omega=2^j\cdot x_1$. As before, by Proposition 2 we have a diffeomorphism $f_j: B_1\# B_2\to B_1\# B_2$ for each $1\leqq j<\infty$. Let $g: B_1\# B_2\to B_1\# B_2$ be a diffeomorphism which corresponds to the automorphism $x_1, y_1, x_2, y_2\to x_2, y_2, x_1, y_1$. Put $\beta(j)=f_j\cdot g$. Hence we have an orientation preserving diffeomorphism $\alpha(l)$: $A_1\# A_2\to A_1\# A_2$ for each $1<l<\infty$ (resp. $\beta(j): B_1\# B_2\to B_1\# B_2$ for each $1\leqq j<\infty$) such that the inducing automorphism $\alpha(l)_*: H_2(A_1\# A_2)\to H_2(A_1\# A_2)$ (resp. $\beta(j)_*: H_2(B_1\# B_2)\to H_2(B_1\# B_2)$) corresponds to the following matrix;

$$A(l)=\begin{bmatrix} l & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & -l & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\qquad\left(\text{resp. } B(j)=\begin{bmatrix} 0 & -2^j & 1 & 0\\ 0 & -2^j & 0 & 1\\ 1 & -2^{2j-1} & 2^j & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\right)$$

We denote by $N(l)$ (resp. $L(j)$) the manifold obtained by identifying points $(x, 0)$ and $(\alpha(l)\cdot x, 1)$ for $x\in A_1\# A_2$ (resp. $(x, 0)$ and $(\beta(j)\cdot x, 1)$ for $x\in B_1\# B_2$) in $(A_1\# A_2)\times[0,1]$ (resp. $(B_1\# B_2)\times[0,1]$). The projection $(A_1\# A_2)\times[0,1]\to[0,1]$ (resp. $(B_1\# B_2)\times[0,1]\to[0,1]$) induces a fiber map $N(l)\to S^1$ with $A_1\# A_2$ as a fiber (resp. $L(j)\to S^1$ with $B_1\# B_2$ as a fiber). Let CP^2 be the complex projective plane. We denote by $L(-1)$ the manifold obtained by attaching $CP^2\times\{0\}$ and $CP^2\times\{1\}$ in $CP^2\times[0,1]$ by a diffeomorphism which reverses the orientation of the projective line. Let $L(\infty)$ be the product $CP^2\times S^1$.

Lemma (i) *$H_2(N(l))=\boldsymbol{Z}_l+\boldsymbol{Z}_l$ for $1<l<\infty$, $H_2(L(j))=\boldsymbol{Z}_{2^j}+\boldsymbol{Z}_{2^j}$ for $1\leqq j<\infty$, $H_2(L(-1))=\boldsymbol{Z}_2$ and $H_2(L(\infty))=\boldsymbol{Z}$.*

(ii) *$\omega^2(N(l))=0$ for $1<l<\infty$. $\omega^2(L(j))\neq 0$ for $j=-1, 1, 2, \cdots, \infty$.*

Proof. (i) It follows by noting the attachment.

(ii) First note that $i^*(\tau(N(l)))=\tau(A_1\# A_2)\oplus\varepsilon^1$ for $1<l<\infty$, $i^*(\tau(L(j)))=\tau(B_1\# B_2)\oplus\varepsilon^1$ for $j\neq -1, \infty$, $i^*(\tau(L(j)))=\tau(\boldsymbol{CP}^2)\oplus\varepsilon^1$ for $j=-1, \infty$, where i is the inclusion map of $A_1\# A_2$ (resp. $B_1\# B_2$ or $\boldsymbol{CP}^2$) into $N(l)$ (resp. $L(j)$) as a fiber and ε^1 is a trivial line bundle. Then we have $i^*\omega^2(N(l))=\omega^2(A_1\# A_2)$, $i^*\omega^2(L(j))=\omega^2(B_1\# B_2)$ for $j\neq 1, \infty$ and $i^*\omega^2(L(j))=\omega^2(\boldsymbol{CP}^2)$ for $j=-1, \infty$. Since $i^*: H^2(N(l); \boldsymbol{Z}_2)\to H^2(A_1\# A_2; \boldsymbol{Z}_2)$ is injective, $\omega^2(A_1\# A_2)$

$=0$, $\omega^2(B_1 \# B_2)\neq 0$ and $\omega^2(CP^2)\neq 0$, we have $\omega^2(N(l))=0$ and $\omega^2(L(j))\neq 0$.

Let $p\in A_1\# A_2$ (resp. $B_1\# B_2$) be a fixed point of $\alpha(l)$ (resp. $\beta(j)$). Let $\varphi: S^1\to N(l)$ (resp. $L(j)$) be an imbedding defined by $t\in[0,1]\to(p,t)\in(A_1\# A_2)\times[0,1]$ (resp. $(B_1\# B_2)\times[0,1]$). This imbedding is transverse to the fibers. Therefore this imbedding is transverse to the foliation on $N(l)$ (resp. $L(j)$) induced from the pointwise foliation of S^1. Then by modifying the foliation on $N(l)$ (resp. $L(j)$), we can obtain the foliation on $N(l)$ (resp. $L(j)$) which contains a *Reeb component* (see [3]). We denote by $(M(l),\partial M(l))$, (resp. $(K(j),\partial K(j))$ the foliated manifold with boundary obtained by removing the *Reeb component* from $N(l)$ (resp. $L(j)$). Then $\partial M(l)$ (resp. $\partial K(j)$) is a closed leaf diffeomorphic to $S^1\times S^3$, and $H_2(M(l))=Z_l+Z_l$, $H_2(K(j))=Z_{2j}+Z_{2j}$ for $1\leqq j<\infty$, $H_2(K(-1))=Z_2$, $H_2(K(\infty))=Z$, $\omega^2(M(l))=0$ and $\omega^2(K(j))\neq 0$.

3. Theorem *Every simply connected, closed 5-manifold admits a codimension 1 foliation.*

Proof. It is sufficient to prove for the case $i(M)\neq 0$ since N. A'Campo [1] has shown the theorem for the case $i(M)=0$. Let M be a simply connected, closed 5-manifold with $i(M)=j$. As first consider for the case $j\neq -1,\infty$. Then we have $H_2(M)=\overbrace{Z+\cdots\cdots+Z}^{n}+Z_{2j}+Z_{2j}+Z_{n_1}+Z_{n_2}+\cdots+Z_{n_s}+Z_{n_s}$. By the way, we have already known in [3] that S^5 admits a codimension 1 foliation. By modifying the foliation on S^5, we can obtain the foliation on S^5 which contains $(n+s+1)$-*Reeb components*. We remove $(n+s+1)$-*Reeb components* from the foliated 5-sphere. Then the resulting manifold is the foliated manifold with $(n+s+1)$-copies of $S^1\times S^3$ as a boundary. We denote it by $(B(n+s+1),\partial B(n+s+1))$. Let X be the manifold obtained by attaching, along the boundaries, a union of n-copies of $(B(1),\partial B(1))$, $(K(j),\partial K(j))$ and $\bigcup_{i=1}^{s}(M(n_i),\partial M(n_i))$ to $(B(n+s+1),\partial B(n+s+1))$. By using Van Kampen's theorem and Mayer Vietoris exact sequence, we can show $\pi_1(X)=0$, $H_2(X)=H_2(M)$, and $i(X)=j$. Therefore X is diffeomorphic to M by Proposition 1. Hence it follows that M admits a codimension 1 foliation. It is similar for the case $j=-1,\infty$.

References

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