12. *A Characterization of Nonstandard Real Fields*

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(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 12, 1974)

Throughout this note, $(R, 0, 1, +, \cdot, \leq)$, or simply R, denotes the ordered field of real numbers, and $\hat{R}$ the union of all sets R_n defined inductively by $R_0 = R$ and $R_{n+1} = \mathscr{P}(\bigcup_{i=0}^{n} R_i)$ $(n = 0, 1, 2, \cdots)$, where $\mathscr{P}(X)$ denotes the power set of X. Let $\mathcal{U}$ be a δ-incomplete ultrafilter on an infinite set I. A *nonstandard real number* is defined to be an individual of the ultrapower of $\hat{R}$ with respect to $\mathcal{U}$, and the set $^*R$ of all nonstandard real numbers to be the value at R_0 of the mapping $a \mapsto {}^*a$ of $\hat{R}$ into $\hat{R}^I$ defined by $^*a(t) = a$ for all $t \in I$, where $=$ and $\in$ in $\hat{R}^I$ are defined for $\boldsymbol{a}, \boldsymbol{b} \in \hat{R}^I$ as follows: $\boldsymbol{a} = \boldsymbol{b}$ if and only if $\{t \in I : \boldsymbol{a}(t) = \boldsymbol{b}(t)\} \in \mathcal{U}$, and $\boldsymbol{a} \in \boldsymbol{b}$ if and only if $\{t \in I : \boldsymbol{a}(t) \in \boldsymbol{b}(t)\} \in \mathcal{U}$. Then as is known*), $(^*R, {}^*0, {}^*1, {}^*+, {}^*\cdot, {}^*\leq)$ is a totally ordered field which will be referred in this note as the *$\mathcal{U}$-nonstandard real field.* Let I be a set. By *nonstandard real field over I* we mean a totally ordered field which is isomorphic to some $\mathcal{U}$-nonstandard real field for a δ-incomplete ultrafilter $\mathcal{U}$ on I.

The purpose of this note is to state a condition characterizing nonstandard real fields among totally ordered fields.

Theorem 1. *A totally ordered field K is a nonstandard real field over a set I if and only if it is non-Archimedean and is a homomorphic image of R^I, the ring of all real valued functions on I with the pointwise addition and the pointwise multiplication.*

This result offers of course an axiom system for a nonstandard real field: *A nonstandard real field over a set I is defined to be any non-Archimedean totally ordered field K containing a complete Archimedean subfield R_0 such that K is a homomorphic image of the ring R_0^I.*

Let K be a totally ordered field. An element x of K is said to be *infinitely large* if $a < x$ for every rational element $a \in K$. Let I be a set. For each real number a, let $^*a$ denote the constant mapping on I defined by $^*a(t) = a$ for all $t \in I$. The ordering $\leq$ on the ring R^I is defined as follows: $\boldsymbol{a} \leq \boldsymbol{b}$ if and only if $\boldsymbol{a}(t) \leq \boldsymbol{b}(t)$ for all $t \in I$.

Proof of Theorem 1. It suffices to prove the "if" part. Let φ be the homomorphism of the ring R^I onto K, that is, φ is a mapping of R^I onto K such that $\varphi(\boldsymbol{a}+\boldsymbol{b}) = \varphi(\boldsymbol{a}) + \varphi(\boldsymbol{b})$ and $\varphi(\boldsymbol{ab}) = \varphi(\boldsymbol{a})\varphi(\boldsymbol{b})$ for all

*) See for example, W. A. J. Luxemburg: What is nonstandard analysis. Amer. Math. Monthly, 80, 38–67 (1973).

$\boldsymbol{a}, \boldsymbol{b} \in R^I$. Obviously $\varphi(*0)=0$, $\varphi(*1)=1$, and $\varphi(-\boldsymbol{a})=-\varphi(\boldsymbol{a})$ for every $\boldsymbol{a} \in R^I$. Moreover, as can readily be seen, $\varphi(\boldsymbol{a}^{-1})=\varphi(\boldsymbol{a})^{-1}$ if $\boldsymbol{a} \in R^I$ is regular, i.e., if $\boldsymbol{a}(t) \neq 0$ for all $t \in I$. Hence $\varphi(*a) \neq 0$ for every non-zero $a \in R$, and so the restriction of φ to the set $R'=\{*a \in R^I : a \in R\}$ is an injection of R' onto $\varphi(R')$. It follows that $\varphi(R')$ is a copy of the real number field R. On the other hand, $\boldsymbol{a}, \boldsymbol{b} \in R^I$ and $\boldsymbol{a} \leq \boldsymbol{b}$ imply $\varphi(\boldsymbol{a}) \leq \varphi(\boldsymbol{b})$; for, letting

$$\boldsymbol{c}(t)=\begin{cases}\sqrt{(\boldsymbol{b}-\boldsymbol{a})(t)} & \text{if } \boldsymbol{a}(t)<\boldsymbol{b}(t),\\ 0 & \text{if } \boldsymbol{a}(t)=\boldsymbol{b}(t),\end{cases}$$

we have $\varphi(\boldsymbol{b})-\varphi(\boldsymbol{a})=\varphi(\boldsymbol{b}-\boldsymbol{a})=\varphi(\boldsymbol{c})^2>0$ because $\boldsymbol{b}-\boldsymbol{a}=\boldsymbol{c}^2$.

Since K is non-Archimedean, there exists an infinitely large element $x \in K$. The surjectivity of φ ensures the existence of an $\boldsymbol{x} \in R^I$ with $\varphi(\boldsymbol{x})=x$. Now if $a \in R$, then we have $\varphi(*a)<\varphi(\boldsymbol{x})$, which implies, by what we have shown above, that $\boldsymbol{x} \leq *a$ does not hold, or equivalently that $a<\boldsymbol{x}(t)$ for some $t \in I$. Thus I is an infinite set.

Let $S^\wedge$ denote the characteristic function of $S \subset I$, that is $S^\wedge(S)=\{1\}$ and $S^\wedge(S^c)=\{0\}$, where S^c is the complement of S in I. We shall prove that $\mathcal{U}=\{S \in \mathcal{P}(I) : \varphi(S^\wedge)=1\}$ is a δ-incomplete ultrafilter on I. Since $\varphi(I^\wedge)=\varphi(*1)=1$ and $\varphi(\emptyset^\wedge)=\varphi(*0)=0$, we have $\mathcal{U} \neq \emptyset$ and $\emptyset \notin \mathcal{U}$. If $S, T \in \mathcal{U}$, then since $(S \cap T)^\wedge=S^\wedge \cdot T^\wedge$, we have $\varphi((S \cap T)^\wedge)=\varphi(S^\wedge \cdot T^\wedge)=\varphi(S^\wedge)\varphi(T^\wedge)=1$, and hence $S \cap T \in \mathcal{U}$. If $S \in \mathcal{U}$ and $S \subset T \subset I$, then $S^\wedge \leq T^\wedge \leq *1$, and so we have $1=\varphi(S^\wedge) \leq \varphi(T^\wedge) \leq \varphi(*1)=1$, which shows that T is in $\mathcal{U}$. Moreover let S be a subset of I. Then since

$$\varphi(S^\wedge)\varphi(S^{c\wedge})=\varphi(S^\wedge \cdot S^{c\wedge})=\varphi(*0)=0 \quad \text{and}$$
$$\varphi(S^\wedge)+\varphi(S^{c\wedge})=\varphi(S^\wedge+S^{c\wedge})=\varphi(*1)=1,$$

it follows that one of $\varphi(S^\wedge)$, $\varphi(S^{c\wedge})$ is 0 and the other is 1. Hence either $S \in \mathcal{U}$ or $S^c \in \mathcal{U}$. Thus $\mathcal{U}$ is an ultrafilter on I. To prove that $\mathcal{U}$ is δ-incomplete, let $\boldsymbol{x}$ be an element of R^I such that $\varphi(\boldsymbol{x})$ is infinitely large, and let $S_n=\{t \in I : n \leq \boldsymbol{x}(t)\}$ for each positive integer n. Then since $\boldsymbol{x} \cdot S_n^{c\wedge} \leq *n$, we have

$$\begin{aligned}\varphi(\boldsymbol{x})&=\varphi(\boldsymbol{x})\varphi(S_n^\wedge+S_n^{c\wedge})=\varphi(\boldsymbol{x})\varphi(S_n^\wedge)+\varphi(\boldsymbol{x}\cdot S_n^{c\wedge})\\ &\leq \varphi(\boldsymbol{x})\varphi(S_n^\wedge)+\varphi(*n)<\varphi(\boldsymbol{x})\varphi(S_n^\wedge)+\varphi(\boldsymbol{x}),\end{aligned}$$

and so we have $0<\varphi(\boldsymbol{x})\varphi(S_n^\wedge)$, which implies $\varphi(S_n^\wedge)=1$. Hence $S_n \in \mathcal{U}$ for every positive integer n. But then for each $t \in I$, there is a positive integer n such that $\boldsymbol{x}(t)<n$. This shows that the intersection of all S_n's is empty. Thus $\mathcal{U}$ is δ-incomplete.

We shall now proceed to prove that the $\mathcal{U}$-nonstandard real field $(*R, *0, *1, *+*\cdot, *\leq)$ is isomorphic to K. Let $\boldsymbol{x} \in *R$. Then there exists a unique $f(\boldsymbol{x}) \in K$ such that $\boldsymbol{x}_0 \in R^I$ and $\boldsymbol{x}_0=\boldsymbol{x}$ in $\hat{R}^I$ imply $\varphi(\boldsymbol{x}_0)=f(\boldsymbol{x})$. In fact, let

$$\boldsymbol{z}(t)=\begin{cases}\boldsymbol{x}(t) & \text{if } \boldsymbol{x}(t) \in R,\\ 0 & \text{otherwise,}\end{cases}$$

and define $f(\boldsymbol{x})=\varphi(\boldsymbol{z})$. If $\boldsymbol{x}_0\in R^I$ and $\boldsymbol{x}_0=\boldsymbol{x}$ in $\hat{R}^I$, then the set $S=\{t\in \boldsymbol{I}: \boldsymbol{z}(t)=\boldsymbol{x}_0(t)\}$ contains the intersection of the sets $\{t\in \boldsymbol{I}: \boldsymbol{x}(t)\in R\}$ and $\{t\in \boldsymbol{I}: \boldsymbol{x}_0(t)=\boldsymbol{x}(t)\}$ which are members of $\mathcal{U}$, and so $S\in\mathcal{U}$. Since $\varphi(S^\wedge)=1$ and $(\boldsymbol{z}-\boldsymbol{x}_0)\cdot S^\wedge={}^*0$, we have

$$\varphi(\boldsymbol{x}_0)=\varphi(\boldsymbol{x}_0)+\varphi({}^*0)=\varphi(\boldsymbol{x}_0)+\varphi((\boldsymbol{z}-\boldsymbol{x}_0)\cdot S^\wedge)$$
$$=\varphi(\boldsymbol{x}_0)+\varphi(\boldsymbol{z}-\boldsymbol{x}_0)\cdot 1=\varphi(\boldsymbol{z})=f(\boldsymbol{x}).$$

The uniqueness of such an $f(\boldsymbol{x})$ follows from the existence of an $\boldsymbol{x}_0\in R^I$ with $\boldsymbol{x}_0=\boldsymbol{x}$ in $\hat{R}^I$, which is ensured by the fact that the set $\{t\in \boldsymbol{I}: \boldsymbol{x}(t)\in R\}$ belongs to $\mathcal{U}$. Thus f is a mapping of ${}^*R$ into K.

If $x\in K$, then $\varphi(\boldsymbol{x})=x$ for some $\boldsymbol{x}\in R^I$, and hence we have $f(\boldsymbol{x})=\varphi(\boldsymbol{x})=x$, which establishes the surjectivity of f.

We claim now that if $\boldsymbol{a},\boldsymbol{b}\in R^I$, then $\varphi(\boldsymbol{a})=\varphi(\boldsymbol{b})$ if and only if $\{t\in \boldsymbol{I}: \boldsymbol{a}(t)=\boldsymbol{b}(t)\}\in\mathcal{U}$. To prove this, it will suffice to show that $\varphi(\boldsymbol{a})=0$ if and only if $S=\{t\in \boldsymbol{I}: \boldsymbol{a}(t)=0\}$ does belong to $\mathcal{U}$. To prove the "only if" part of this statement, consider an element $\boldsymbol{b}\in R^I$ defined by

$$\boldsymbol{b}(t)=\begin{cases}\boldsymbol{a}(t)^{-1} & \text{if } t\notin S,\\ 0 & \text{if } t\in S.\end{cases}$$

Then we have $\varphi(S^{c\wedge})=\varphi(\boldsymbol{a}\boldsymbol{b})=\varphi(\boldsymbol{a})\varphi(\boldsymbol{b})=0$, which shows that S is in $\mathcal{U}$. The "if" part of this statement follows immediately from the fact that $\boldsymbol{a}\cdot S^\wedge={}^*0$; i.e. $\varphi(\boldsymbol{a})=\varphi(\boldsymbol{a})\varphi(S^\wedge)=\varphi(\boldsymbol{a}\cdot S^\wedge)=\varphi({}^*0)=0$.

In order to prove that f is an injection, suppose that $\boldsymbol{x},\boldsymbol{y}\in{}^*R$ and $f(\boldsymbol{x})=f(\boldsymbol{y})$. Then we can find $\boldsymbol{x}_0,\boldsymbol{y}_0\in R^I$ such that $\boldsymbol{x}_0=\boldsymbol{x}$ and $\boldsymbol{y}_0=\boldsymbol{y}$ in $\hat{R}^I$. Since $\varphi(\boldsymbol{x}_0)=f(\boldsymbol{x})=f(\boldsymbol{y})=\varphi(\boldsymbol{y}_0)$, the set $S=\{t\in \boldsymbol{I}: \boldsymbol{x}_0(t)=\boldsymbol{y}_0(t)\}$ belongs to $\mathcal{U}$, and consequently we have $\boldsymbol{x}_0=\boldsymbol{y}_0$ in $\hat{R}^I$, which yields the desired conclusion $\boldsymbol{x}=\boldsymbol{y}$ in $\hat{R}^I$.

Suppose that $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in{}^*R$ and $\boldsymbol{x}\ {}^*{+}\,\boldsymbol{y}=\boldsymbol{z}$. Then there exist $\boldsymbol{x}_0,\boldsymbol{y}_0,\boldsymbol{z}_0\in R^I$ such that $\boldsymbol{x}_0=\boldsymbol{x},\boldsymbol{y}_0=\boldsymbol{y}$ and $\boldsymbol{z}_0=\boldsymbol{z}$ in $\hat{R}^I$. Since the sets $\{t\in \boldsymbol{I}: \boldsymbol{x}(t)+\boldsymbol{y}(t)=\boldsymbol{z}(t)\}$, $\{t\in \boldsymbol{I}: \boldsymbol{x}_0(t)=\boldsymbol{x}(t)\}$, $\{t\in \boldsymbol{I}: \boldsymbol{y}_0(t)=\boldsymbol{y}(t)\}$ and $\{t\in \boldsymbol{I}: \boldsymbol{z}_0(t)=\boldsymbol{z}(t)\}$ belong to $\mathcal{U}$, so does their intersection S. But then the set $T=\{t\in \boldsymbol{I}: (\boldsymbol{x}_0+\boldsymbol{y}_0)(t)=\boldsymbol{z}_0(t)\}$ contains S, and hence T is a member of $\mathcal{U}$. Therefore we have $\varphi(\boldsymbol{x}_0+\boldsymbol{y}_0)=\varphi(\boldsymbol{z}_0)$ as is shown above. Consequently we obtain

$$f(\boldsymbol{x}\ {}^*{+}\,\boldsymbol{y})=f(\boldsymbol{z})=\varphi(\boldsymbol{z}_0)=\varphi(\boldsymbol{x}_0+\boldsymbol{y}_0)=\varphi(\boldsymbol{x}_0)+\varphi(\boldsymbol{y}_0)$$
$$=f(\boldsymbol{x})+f(\boldsymbol{y}).$$

A similar argument establishes $f(\boldsymbol{x}\ {}^*{\cdot}\,\boldsymbol{y})=f(\boldsymbol{x})f(\boldsymbol{y})$ for every $\boldsymbol{x},\boldsymbol{y}\in{}^*R$.

Now suppose that $\boldsymbol{x},\boldsymbol{y}\in{}^*R$ and $\boldsymbol{x}\ {}^*{\leq}\,\boldsymbol{y}$. Then there exists a $\boldsymbol{z}_0\in R^I$ such that ${}^*0\ {}^*{\leq}\,\boldsymbol{z}_0=\boldsymbol{y}\ {}^*{+}\,(-\boldsymbol{x})$ in $\hat{R}^I$. Hence $S=\{t\in \boldsymbol{I}: 0\leq\boldsymbol{z}_0(t)\}\in\mathcal{U}$ and ${}^*0\leq\boldsymbol{z}_0\cdot S^\wedge$ in R^I. Therefore we have

$$0=\varphi({}^*0)\leq\varphi(\boldsymbol{z}_0\cdot S^\wedge)=\varphi(\boldsymbol{z}_0)\varphi(S^\wedge)=\varphi(\boldsymbol{z}_0)=f(\boldsymbol{y}\ {}^*{+}\,(-\boldsymbol{x})),$$

which implies $f(\boldsymbol{x})\leq f(\boldsymbol{y})$ because $f(-\boldsymbol{x})=-f(\boldsymbol{x})$. This completes the proof.

In the above theorem and definition, the condition that K is a homomorphic image of R^I cannot be eliminated. To establish this, we need the following

Lemma. *Let K be a non-Archimedean totally ordered field containing a complete Archimedean subfield R_0. If x is an infinitely large element of K, then $\sum_{i=0}^{n} a_i x^i < x^{n+1}$ for every $a_0, a_1, \cdots, a_n \in R_0$, where x^0 denotes the unit element 1 of K.*

Proof. If $a \in R_0$ then we have $ax^{n+1} < x^{n+2}$, since $a < x$ and $0 < x^{n+1}$. Now the assertion of the lemma is trivial if $n=0$. Suppose that it holds for a non-negative integer n, and let $a_0, a_1, \cdots, a_{n+1} \in R_0$. Then we have

$$\sum_{i=0}^{n+1} a_i x^i = \sum_{i=0}^{n} a_i x^i + a_{n+1} x^{n+1} < x^{n+1} + a_{n+1} x^{n+1} = (1 + a_{n+1}) x^{n+1} < x^{n+2}.$$

Corollary. *Let K be a non-Archimedean totally ordered field containing a complete Archimedean subfield R_0. Then each infinitely large element x of K is transcendential relative to R_0.*

Proof. Assume that $\sum_{i=0}^{n} a_i x^i = 0$ $(a_i \in R_0)$ implies $a_i = 0$ for every $i \in \{0, 1, \cdots, n\}$. If $\sum_{i=0}^{n+1} a_i x^i = 0$ $(a_i \in R_0)$ and if $a_{n+1} \neq 0$, then we have $x^{n+1} = -\sum_{i=0}^{n} a_i a_{n+1}^{-1} x^i$, contrary to Lemma. Hence if $\sum_{i=0}^{n+1} a_i x^i = 0$ $(a_i \in R_0)$, then $a_{n+1} = 0$, and so $a_0 = a_1 = \cdots = a_n = 0$.

We shall now prove the following

Theorem 2. *There exists a non-Archimedean totally ordered field K containing a complete Archimedean subfield R_0 such that K is not a nonstandard real field over any set.*

Proof. Let x be an infinitely large element of a nonstandard real field $^*R$ over some set, and let R_0 be the subfield of all standard numbers of $^*R$. R_0 is a complete Archimedean subfield of $^*R$. Let us denote by K the smallest subfield of $^*R$ containing $R_0 \cup \{x\}$, and suppose that K is a nonstandard real field over a set I. Then K is isomorphic to some $\mathcal{U}$-nonstandard real field for a δ-incomplete ultrafilter $\mathcal{U}$ on I. We identify K with this $\mathcal{U}$-nonstandard real field. Let S be the set of all $t \in I$ with $x(t) \in R$, and let

$$a(t) = \begin{cases} \sqrt{x(t)} & \text{if } t \in S, \\ 0 & \text{if } t \notin S. \end{cases}$$

Then since $I, S \in \mathcal{U}$ and $S \subset \{t \in I : a^2(t) = x(t)\}$, it follows that $a \in K$ and $a^2 = x$. Consequently we can find $a_0, \cdots, a_m, b_0, \cdots, b_n \in R_0$ with $a_m \neq 0$ and $b_n \neq 0$ such that

$$a = \left(\sum_{i=0}^{m} a_i x^i\right)\left(\sum_{i=0}^{n} b_i x^i\right)^{-1},$$

and hence

$$x\left(\sum_{i=0}^{n} b_i x^i\right)^2 - \left(\sum_{i=0}^{m} a_i x^i\right)^2 = 0,$$

where x^0 denotes the unit element 1 of K. Thus if $2n+1 > 2m$, then by the above Corollary, we have a contradiction $b_n^2 = 0$; if $2n+1 \leq 2m$, then since $2n+1 < 2m$, the same Corollary yields a contradiction $a_m^2 = 0$. This completes the proof.