30. *A Necessary Condition for the Well-Posedness of the Cauchy Problem for a Certain Class of Evolution Equations*

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§ 1. **Introduction.** We consider the Cauchy problem for an evolution equation

$$(*)\qquad \begin{cases}(\partial_t - i\partial_x^2 - b(x,t)\partial_x)u(x,t)=0, & (x,t)\in \boldsymbol{R}^1\times[0,T],\\ u(x,0)=u_0(x),\end{cases}$$

where

$$b(x,t)\in \mathcal{E}_t^0(\mathcal{B}^\infty),\quad u_0(x)\in \mathcal{D}_{L^2}^\infty,\quad \partial_t=\frac{\partial}{\partial t},\quad \partial_x=\frac{\partial}{\partial x}.$$

Under what conditions is the Cauchy problem $(*)$ well posed?

In the case where $b(x,t)$ is constant, Hadamard's condition shows that the necessary and sufficient condition for the Cauchy problem $(*)$ to be well posed is that the coefficient b is a real number (see Theorem 5.3 in S. Mizohata [2]). In the case where $b(x,t)$ is a real-valued function, it is easy to see that the Cauchy problem $(*)$ is well posed in $\mathcal{D}_{L^2}^\infty$. In the case where $\mathcal{I}m\, b(x,t)\not\equiv 0$, as we shall see below, the situation is much more delicate. In order to make this situation clear, we assume that $b(x,t)$ is a function depending only on x, denote it by $b(x)$:

$$(**)\qquad \begin{cases}(\partial_t - i\partial_x^2 - b(x)\partial_x)u(x,t)=0 & (x,t)\in \boldsymbol{R}^1\times[0,T],\\ u(x,0)=u_0(x).\end{cases}$$

As we mentioned above, if we fix x_0 such that $\mathcal{I}m\, b(x_0)\neq 0$, then the Cauchy problem for the tangential operator (i.e. operator freezing the coefficients) $\partial_t - i\partial_x^2 - b(x_0)\partial_x$ is not well posed in $\mathcal{D}_{L^2}^\infty$. But in the case where the coefficients depend on x, the situation is different. The following assertion holds:

Assume that $\mathcal{I}m\, b(x)$ belongs to $L^1(\boldsymbol{R}^1)\cap\mathcal{B}^\infty$. Then the Cauchy problem $(**)$ is well posed in $\mathcal{D}_{L^2}^\infty$.

To see this, it is sufficient to note that the linear mapping

$$\mathcal{E}_t^1(\mathcal{D}_{L^2}^\infty)\ni u(x,t)\to v(x,t)=u(x,t)\exp\left(\frac{1}{2}\int_{-\infty}^x \mathcal{I}m\, b(y)dy\right)\in \mathcal{E}_t^1(\mathcal{D}_{L^2}^\infty)$$

is one-to-one, onto, continuous and that $v(x,t)$ satisfies the equation

$$(***)\qquad \begin{cases}(\partial_t - i\partial_x^2 - \mathcal{R}e\, b(x)\partial_x + c(x))v(x,t)=0,\\ v(x,0)=u_0(x)\exp\left(\dfrac{1}{2}\displaystyle\int_{-\infty}^x \mathcal{I}m\, b(y)dy\right),\end{cases}$$

where $c(x)=\frac{i}{2}(\mathcal{I}m\, b(x))'+\frac{i}{4}(\mathcal{I}m\, b(x))^2+\frac{1}{2}(\mathcal{R}e\, b(x))(\mathcal{I}m\, b(x))$, and that the Cauchy problem $(***)$ is well posed in $\mathcal{D}^\infty_{L^2}$.

On the other hand, *suppose that* $|\mathcal{I}m\, b(x)|\geqq\delta>0$ *for all* $x\in \boldsymbol{R}^1$, *then the Cauchy problem* $(**)$ *is not well posed in* $\mathcal{D}^\infty_{L^2}$ (see the following theorem).

Let

(1.1) $$\partial_t u(x,t)-a(x,t\,;D)u(x,t)=0$$

be an evolution equation defined on $(x,t)\in \boldsymbol{R}^l\times[0,T]$ where

$$a(x,t\,;D)=\sum_{j=0}^{m}a_j(x,t\,;D),$$

$$a_j(x,t\,;D)=\sum_{|\nu|=j}a_\nu(x,t)D^\nu,\qquad a_\nu(x,t)\in\mathcal{E}^0_t(\mathcal{B}^\infty),$$

$$D=\left(-i\frac{\partial}{\partial x_1},\cdots,-i\frac{\partial}{\partial x_l}\right),\qquad D^\nu=\left(-i\frac{\partial}{\partial x_1}\right)^{\nu_1}\cdots\left(-i\frac{\partial}{\partial x_l}\right)^{\nu_l},$$

$\nu=(\nu_1,\cdots,\nu_l)$ is multi-index of non-negative integers and $|\nu|=\nu_1+\cdots+\nu_l$.

We are concerned with the Cauchy problem for (1.1).

Our purpose of this article is to prove the following

Theorem. *Suppose that there exists an integer* p $(1\leqq p\leqq m-1)$ *such that the following conditions hold*:

(C1) $a_m(x,t\,;D),\cdots,a_{p+1}(x,t\,;D)$ *are differential operators whose coefficients are independent of* x. *Denote* $a_j(x,t\,;D)$ *by* $a_j(t\,;D)$ *for* $p+1\leqq j\leqq m$.

(C2) $\mathcal{R}e\, a_j(t\,;\xi)\equiv 0$ *for* $(t\,;\xi)\in[0,T]\times \boldsymbol{R}^l$, $p+1\leqq j\leqq m$.

(C3) *there exist* $\xi_0\in S^{l-1}_\xi=\{\xi\in \boldsymbol{R}^l\,;|\xi|=1\}$ *and* $t_0\in[0,T)$ *satisfying*

(1.2) $$\inf_{x\in \boldsymbol{R}^l}\mathcal{R}e\, a_p(x,t_0\,;\xi_0)>0.$$

Then the forward Cauchy problem for (1.1) *with initial data at* $t=t_0$ *is not well posed in* $\mathcal{D}^\infty_{L^2}$ *in any small neighborhood of* $t=t_0$.

This theorem is proved by the localization of operator and energy inequalities whose method was developed by S. Mizohata [1] (see also I. G. Petrowsky [3]).

§ 2. Localization of the operator $a_p(x,t\,;D)$. Condition (C3) implies that there exist $T_0(>t_0)$, $\delta_1>0$ and a neighborhood $V(\xi_0)$ of ξ_0 such that

(2.1) $$\mathcal{R}e\, a_p(x,t\,;\xi)\geqq\delta_1\qquad\text{for } (x,t\,;\xi)\in \boldsymbol{R}^l\times[t_0,T_0]\times V(\xi_0).$$

We can choose $\varepsilon>0$ such that

$$U_{4\varepsilon}(\xi_0)=\{\xi\,;|\xi-\xi_0|<4\varepsilon\}\subset V(\xi_0).$$

Define $\alpha(\xi)\in C^\infty_0(\boldsymbol{R}^l)$ such that supp $[\alpha(\xi)]\subset U_{2\varepsilon}(\xi_0)$, $\alpha(\xi)=1$ on $U_\varepsilon(\xi_0)$ and $0\leqq\alpha(\xi)\leqq 1$. We put

(2.2) $$\alpha_n(\xi)=\alpha(\xi/n)$$

and define convolution operators $\alpha_n(D)$ and $\alpha_n^{(\nu)}(D)$ as follows:

$$\begin{aligned}\alpha_n(D)u(x)&=\mathcal{F}^{-1}[\alpha_n(\xi)\hat{u}(\xi)],\\ \alpha_n^{(\nu)}(D)u(x)&=\mathcal{F}^{-1}[\alpha_n^{(\nu)}(\xi)\hat{u}(\xi)],\end{aligned}\tag{2.3}$$

where

$$\alpha_n^{(\nu)}(\xi)=\left(\frac{\partial}{\partial\xi}\right)^{\nu}\alpha_n(\xi).$$

We take a C^∞-mapping θ from S_ξ^{l-1} to S_ξ^{l-1} such that

i) $\theta(\xi')\in U_{4\varepsilon}(\xi_0)\cap S_\xi^{l-1}$, $(\xi'=\xi/|\xi|)$,

ii) $\theta(\xi')=\xi'$ on $U_{3\varepsilon}(\xi_0)$.

For any $\xi\in \boldsymbol{R}^l$, we define $\theta(\xi)=\theta(\xi')|\xi|$.

Define a pseudo-differential operator $\tilde{a}_p(x,t;D)$ whose symbol is

$$\tilde{a}_p(x,t;\xi)=a_p(x,t;\theta(\xi)).\tag{2.4}$$

Then we have

$$\tilde{a}_p(x,t;D)(\alpha_n(D)u)=a_p(x,t;D)(\alpha_n(D)u).\tag{2.5}$$

By the construction of $\tilde{a}_p(x,t;\xi)$, we have

$$\mathcal{R}e\,\tilde{a}_p(x,t;\xi)\geqq\delta_1|\xi|^p\qquad\text{for }(x,t;\xi)\in\boldsymbol{R}^l\times[t_0,T_0]\times\boldsymbol{R}^l,\tag{2.6}$$

$$\mathcal{R}e\,(\tilde{a}_p(x,t;D)(\alpha_n(D)u),\alpha_n(D)u)\geqq\delta_2 n^p\|\alpha_n u\|^2\qquad(\delta_2>0).\tag{2.7}$$

§ 3. Energy inequality. Applying $\alpha_n(D)$ to (1.1), we have

$$\partial_t(\alpha_n(D)u)=a(x,t;D)(\alpha_n(D)u)+[\alpha_n(D),a(x,t;D)]u.\tag{3.1}$$

From this equation we obtain the following

Lemma. *For $u(x,t)$ satisfying* (1.1), *the energy inequality*

$$\frac{d}{dt}\|\alpha_n(D)u\|^2\geqq\delta_3 n^p\|\alpha_n u\|^2-Cn^p\sum_{1\le|\nu|\le k}\|\alpha_n^{(\nu)}(D)u\|^2-Cn^{p-2(k+1)}\|u\|^2\tag{3.2}$$

holds (for n large) where δ_3 is a positive constant independent of n, C is a constant independent of n (from now on we denote various constants independent of n by C) and where $\|\cdot\|$ is $L^2(\boldsymbol{R}_x^l)$-norm. More generally, for $|\nu|\leqq k$, we have

$$\begin{aligned}\frac{d}{dt}\|\alpha_n^{(\nu)}(D)u\|^2\geqq\delta_3 n^p\|\alpha_n^{(\nu)}(D)u\|^2-Cn^p\sum_{|\nu|+1\le|\nu'|\le k}\|\alpha_n^{(\nu')}u\|^2\\ -Cn^{p-2(k+1)}\|u\|^2.\end{aligned}\tag{3.3}$$

Proof. In view of (C2), (2.5) and (2.7), from (3.1) we have

$$\begin{aligned}\frac{d}{dt}\|\alpha_n(D)u\|^2&=2\,\mathcal{R}e\,(a(x,t;D)(\alpha_n u),\alpha_n u)+2\,\mathcal{R}e\,([\alpha_n,a]u,\alpha_n u)\\ &\geqq\frac{3}{4}\delta_2 n^p\|\alpha_n(D)u\|^2-2\|\alpha_n u\|\cdot\|[\alpha_n,a]u\|.\quad(\text{for } n \text{ large})\end{aligned}$$

$$\frac{d}{dt}\|\alpha_n u\|^2\geqq\frac{1}{2}\delta_2 n^p\|\alpha_n(D)u\|^2-\frac{4}{\delta_2}n^{-p}\|[\alpha_n,a]u\|^2.\tag{3.4}$$

Now, we shall estimate the commutator term $[\alpha_n,a]u$.

Expanding the commutator, we have

$$[\alpha_n,a]u=\sum_{1\le|\nu|\le k}\frac{1}{\nu!}D_x^\nu a(x,t;D)\alpha_n^{(\nu)}(D)u+R_k(u),\tag{3.5}$$

where $D_x^\nu a(x,t;D)$ is a differential operator whose symbol is $D_x^\nu a(x,t;\xi)$.

In view of (C1), the order of $D_x^\nu a(x,t;D)$ is p, thus we have

(3.6) $$\|R_k(u)\| \leqq Cn^{p-(k+1)} \cdot \|u\|.$$

From (3.5) and (3.6), we have

(3.7) $$\|[\alpha_n, a]u\|^2 \leqq Cn^{2p} \sum_{1\leq|\nu|\leq k} \|\alpha_n^{(\nu)}(D)u\|^2 + Cn^{2p-2(k+1)}\|u\|^2.$$

(3.2) follows from (3.4) and (3.7).

Replacing $\alpha_n(D)$ by $\alpha_n^{(\nu)}(D)$, we obtain the inequality (3.3).

§ 4. **Proof of the theorem.** Suppose that the Cauchy problem for (1.1) with initial data at $t=t_0$ is well posed in $\mathcal{D}_{L^2}^\infty$.

At first, we choose a function $\hat{\psi}(\xi) \in C_0^\infty$ such that the support of $\hat{\psi}(\xi)$ is contained in a neighborhood $U_\epsilon(0)$ of the origin and $\hat{\psi}(\xi) \geqq 0$, $\int \hat{\psi}(\xi)d\xi = 1$. Then $\alpha(\xi)=1$ on the support of $\hat{\psi}(\xi-\xi_0)$. Let us denote $\psi(x) = \mathcal{F}^{-1}[\hat{\psi}(\xi)]$. Define a sequence $u_n(x,t)$ of solutions of (1.1) with initial data

(4.1) $$u_n(x,t_0) = e^{inx\xi_0}\psi(x).$$

By hypothesis, there exist a positive integer h and a positive constant C such that

(4.2) $$\|u_n(t)\| \leqq C\|u_n(t_0)\|_h \leqq C'n^h.$$

We replace $u(x,t)$ in the section 3 by $u_n(x,t)$ and take $k=h$.

Define

(4.3) $$S_n(t) = \sum_{|\nu|=0}^{h} M^{|\nu|}\|\alpha_n^{(\nu)}(D)u_n(t)\|^2 \qquad \text{for sufficiently large } M.$$

From (3.2) and (3.3), we have

(4.4) $$\frac{d}{dt}S_n(t) \geqq \delta n^p S_n(t) - Cn^{p-2},$$

where δ is a positive constant independent of n.

Thus we obtain

(4.5) $$S_n(t) \geqq \left\{S_n(t_0) - \frac{C}{\delta}n^{-2}\right\}e^{\delta n^p(t-t_0)}$$

Lemma. $S_n(t_0) = \|\psi\|^2 > 0$.

Proof. Since $\alpha_n(\xi)=1$ on supp $[\hat{\psi}(\xi-n\xi_0)]$, we have

$$\begin{aligned} S_n(t_0) &= \sum_{|\nu|=0}^{h} M^{|\nu|}\|\alpha_n^{(\nu)}(D)(e^{inx\xi_0}\psi(x))\|^2 \\ &= \sum_{|\nu|=0}^{h} M^{|\nu|}\|\alpha_n^{(\nu)}(\xi)\hat{\psi}(\xi-n\xi_0)\|^2 \\ &= \|\alpha_n(\xi)\hat{\psi}(\xi-n\xi_0)\|^2 + \sum_{1\leq|\nu|\leq h} M^{|\nu|}\|\alpha_n^{(\nu)}(\xi)\hat{\psi}(\xi-n\xi_0)\|^2 \\ &= \|\hat{\psi}(\xi-n\xi_0)\|^2 = \|\psi\|^2 > 0. \qquad \text{(Q.E.D.)} \end{aligned}$$

Finally, we have

(4.6) $$S_n(t) \geqq \delta_0 e^{\delta n^p(t-t_0)} \qquad \text{for large } n$$

where δ_0 and δ are positive constants.

On the other hand, from (4.2) and (4.3), we have

(4.7) $$S_n(t) \leqq Cn^{2h}.$$

For any t ($t_0 < t < T_0$) and large n, (4.6) and (4.7) are not compatible which is contradiction. This completes the proof of the theorem.

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References

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