27. Riemannian Manifolds Admitting Some Geodesic

By Tetsunori KUROGI Fukui University

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1. Introduction. Let M be a compact Riemannian manifold and f an isometry of M. Then a geodesic α on M is called f-invariant geodesic if $f\alpha = \alpha$. It is not known much about isometry invariant geodesic. In this paper we see what kind of Riemannian manifold admits an isometry invariant geodesic. Our results are following;

Theorem A (K. Grove). Let M be a compact connected, simply connected and oriented Riemannian manifold of odd dimension and fan orientation preserving isometry of M. Then there exists an f-invariant geodesic.

Theorem B. Let M be a compact connected, simply connected and oriented Riemannian manifold of 2k-dimension and f an orientation preserving isometry of M. Then there exists an f-invariant geodesic for k=1 and also well for k>1 if $\lambda_k(f)=even$ where $\lambda_k(f)$ is the trace of an induced homomorphism $f_k: H_k(M, Q) \to H_k(M, Q)$ where Q is the field of rational numbers.

Corollary. Let M be a manifold of Theorem B. Then M admits an f-invariant geodesic for any orientation preserving isometry f of M if $H_k(M, Q) = 0$.

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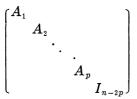
2. Fixed points of isometry. Let M be a compact manifold and f be an isometry of M. Then the induced homomorphism by f of the *i*-th homology group of M over coefficient Q is denoted by $f_i: H_i(M, Q) \to H_i(M, Q)$ and the trace of f_i by $\lambda_i(f)$.

Lemma 1. Let M be an n-dimensional orientable Riemannian manifold and f an orientation preserving isometry, then we have $\lambda_i(f) = \lambda_{n-i}(f)$ $(i=1 \sim n)$.

Proof. We have only to use the Poincaré duality. q.e.d.

Lemma 2. Let M be an odd dimensional orientable Riemannian manifold and f an orientation preserving isometry of M, then f has no isolated fixed points.

Proof. Let x be a fixed point of f and $f_*: T_x(M) \to T_x(M)$ be an induced homomorphism by f. Then f_* is an element of SO(n) and so f_* has a following representation with respect to a suitable basis;



where

$$A_i = \begin{pmatrix} \cos heta_i, & -\sin heta_i \\ \sin heta_i, & \cos heta_i \end{pmatrix}, \quad I_{n-2p} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

If we take a vector $V = (0, 0, \dots, 0, v_{2p+1}, \dots, v_n)$ of $T_x(M)$ with $|V| < \varepsilon$ for any positive ε where $|\cdot|$ is a norm in $T_x(M)$, then exp (V) is a fixed point of f because of $\exp \cdot f_* = f \cdot \exp$. Therefore x is not an isolated fixed point.

For fixed point of isometry the following theorem is very useful;

Theorem C (S. Kobayashi). Let M be a compact Riemannian manifold and f an isometry of M. Let F be the fixed point set of f. If we denote the Lefschetz number of f by L(f) and the Euler number of F by $\chi(F)$, then $L(f) = \chi(F)$.

3. Proof of theorems. K. Grove has obtained the following result in his paper [1].

Theorem D (K. Grove). Let M be a compact connected and simply connected Riemannian manifold. Then for any isometry f of M without or with at least two fixed points M admits an f-invariant geodesic.

By this theorem and Lemma 2 we have Theorem A. Now we go to the proof of Theorem B. The case of k > 1: By Lemma 1 we have $L(f)=2\sum_{i=2}^{k-1}(-1)^i\lambda_i(f)+(-1)^k\lambda_k(f)+2$. Thus if $\lambda_k(f)=$ even, $L(f)\neq 1$. Therefore we have $\chi(F)\neq 1$. If $\chi(F)=0$ there are no fixed points or Fcontains non isolated fixed point set. Hence M admits an f-invariant geodesic by Theorem D. The case of k=1: By assumption of Theorem B we have L(f)=2, therefore $\chi(F)=2$. So F consists of at least two fixed points. Thus by virtue of Theorem D M admits an f-invariant geodesic. This completes the proof of Theorem B. In particular $\lambda_k(f)=0$ is an optimal solution of the equation $L(f)\neq 1$ and so corollary follows.

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References

- [1] Grove, K.: Condition(C) for the energy integral on certain path-spaces and applications to the theory of geodesics (preprint series of Aarhus Universitet (1970)).
- [2] Kobayashi, S.: Transformation groups in differential geometry. Ergebnisse der Math., Bd, 70 (1972).