84. *Extremely Amenable Transformation Semigroups. II*

By Kôkichi SAKAI

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Introduction. Let S be a semigroup and X a nonvoid set. Then we shall say that the pair (S,X) is a *transformation semigroup* if for every $s\in S$ there corresponds a map: $X\ni x\mapsto sx\in X$ such that $s(tx)=(st)x$ for all s,t in S and x in X. Let $B(X)$ be the Banach algebra of all real valued bounded functions on X with the supremum norm and $B(X)^*$ the conjugate Banach space of $B(X)$. For every $s\in S$ define the map $L_s: B(X)\to B(X)$ by $L_s f={}_sf$ for $f\in B(X)$, where ${}_sf(x)=f(sx)$ for x in X. Then we have $L_sL_t=L_{ts}$ and $\|L_s\|\le 1$ for all s,t in S. The map $L: s\mapsto L_s$ is called the left regular antirepresentation of S on $B(X)$. $\varphi\in B(X)^*$ is a *mean* on $B(X)$ if $\inf\{f(x): x\in X\}\le\varphi(f)\le\sup\{f(x): x\in X\}$ for all $f\in B(X)$. If φ is a mean on $B(X)$, we have $\|\varphi\|=\varphi(I_X)=1$ where I_X is the constant one function on X. $\varphi\in B(X)^*$ is called *invariant* if $\varphi({}_sf)=\varphi(f)$ for all $(s,f)\in S\times B(X)$. $\varphi\in B(X)^*$ is *multiplicative* if $\varphi(f\circ g)=\varphi(f)\cdot\varphi(g)$ for all $f,g\in B(X)$. By βX denote the set of all multiplicative means on $B(X)$, which is a w^*-compact subset of $B(X)^*$. For every $x\in X$ define $\delta_x\in\beta X$ by $\delta_x(f)=f(x)$ for all $f\in B(X)$ and denote by δ the map: $X\ni x\mapsto\delta_x\in\beta X$. Now we shall say a transformation semigroup (S,X) is *extremely amenable* if there is a multiplicative invariant mean on $B(X)$.

On extremely amenable transformation semigroups they are investigated by E. Granirer in [2] and by the author in [6]. In this paper, using the results in [2] and [6], we shall give various characterizations of extremely amenable transformation semigroups by means of the so-called *"fixed-point property"*, *"multiplicative invariant extension property"* and *"Reiter-Glicksberg's inequality"*. In § 4 we note addenda to my papers [6] and [7].

§ 1. Fixed-point property. We say a transformation semigroup (S,X) has a *fixed-point* if there is some x_0 in X such that $sx_0=x_0$ for all $s\in S$. A transformation semigroup (S,Z) is called *compact* if Z is a compact Hausdorff space and for every $s\in S$ the map: $Z\ni z\mapsto sz\in Z$ is continuous. For example, for every $(s,\varphi)\in S\times\beta X$ define $s\varphi\in\beta X$ by $s\varphi(f)=\varphi({}_sf)$ for $f\in B(X)$. Then $(S,\beta X)$ is compact. Clearly (S,X) is extremely amenable if and only if $(S,\beta X)$ has a fixed-point. Let (S,X) and (S,Y) be transformation semigroups. A map $\sigma: X\to Y$ is called a *homomorphism* of (S,X) to (S,Y) if $s\sigma(x)=\sigma(sx)$ for all $(s,x)\in S\times X$.

The following theorem is a generalization of the so-called "*common fixed-point property on compacta*" due to T. Mitchell [5].

Theorem 1. *The following properties are equivalent.*

(1) *A transformation semigroup (S, X) is extremely amenable.*

(2) *For any given compact transformation semigroup (S, Z), if there is a homomorphism σ of (S, X) to (S, Z), then (S, Z) has a fixed-point.*

Proof. (1)⇒(2): Let φ be a multiplicative invariant mean on $B(X)$ and $C(Z)$ the Banach algebra of all real valued continuous functions on Z with the supremum norm. For any $f \in C(Z)$ define $\sigma^* f \in B(X)$ by $\sigma^* f(x) = f(\sigma(x))$ for $x \in X$. Then σ^* is a continuous homomorphism of $C(Z)$ into $B(X)$ as Banach algebra and $\tilde{\varphi} = \varphi \cdot \sigma^*$ is a nonzero multiplicative linear functional on $C(Z)$. So there is a point z_0 such that $\tilde{\varphi}(f) = f(z_0)$ for all $f \in C(Z)$. Furthermore we have

$$f(sz_0) = \tilde{\varphi}({}_s f) = \varphi(\sigma^*({}_s f)) = \varphi({}_s(\sigma^* f)) = \varphi(\sigma^*(f)) = \tilde{\varphi}(f) = f(z_0)$$

for every $(s, f) \in S \times C(Z)$. Since $C(Z)$ separates the points of Z, we have $sz_0 = z_0$ for all $s \in S$.

(2)⇒(1): $(S, \beta X)$ is compact and the map δ is a homomorphism of (S, X) to $(S, \beta X)$. So, by (2), $(S, \beta X)$ has a fixed-point. Thus (S, X) is extremely amenable. q.e.d.

§ 2. **Multiplicative invariant extension property.** Let (S, X) be a transformation semigroup, E and F (real or complex) algebras, $\mathrm{Hom}(E, F)$ the set of all homomorphisms of E into F as algebras and $\{T_s; s \in S\}$ an antirepresentation of S by homomorphisms of E into itself. Now, for every $(s, \Phi) \in S \times \mathrm{Hom}(E, F)$ define $s\Phi \in \mathrm{Hom}(E, F)$ by $s\Phi = \Phi T_s$. Then $(S, \mathrm{Hom}(E, F))$ is a transformation semigroup. Furthermore we assume that there exists a homomorphism σ of (S, X) to $(S, \mathrm{Hom}(E, F))$, that is, there is a map $\sigma: X \to \mathrm{Hom}(E, F)$ such that $\sigma(sx) = \sigma(x) T_s$ for all $(s, x) \in S \times X$. In this case we shall say the collection $\{T, E, \sigma, F\}$ is an *algebra representation* of (S, X). Especially if E and F are Banach algebras with unit, the collection $\{T, E, \sigma, F\}$ is called a *Banach algebra representation* of (S, X). For example $\{L, B(X), \delta, R\}$ is a Banach algebra representation of (S, X), where L is the left regular antirepresentation of S on $B(X)$ and R the real field. By $\mathfrak{M}(E)$ denote the set of all multiplicative linear functionals on a Banach algebra E with unit, which is w^*-compact in E^*. We shall say a Banach algebra representation $\{T, E, \sigma, F\}$ of (S, X) has *multiplicative invariant extension property* if it satisfies the following conditions (♯).

(♯): *Let E_0 be any subalgebra of E such that $T_s(E_0) \subseteq E_0$ for all $s \in S$ and $\varphi_0 \in \mathfrak{M}(E_0)$ satisfy $\varphi_0(T_s f) = \varphi_0(f)$ for all $(s, f) \in S \times E_0$. Moreover assume that there exists a $\psi \in \mathfrak{M}(F)$ such that $\varphi_0(f) = \psi(\sigma(x) f)$ for all $(x, f) \in X \times E_0$. Then there exists an extension $\varphi \in \mathfrak{M}(E)$ of φ_0 such* *that* $\varphi(T_s f)=\varphi(f)$ *for all* $(s,f)\in S\times E$.

Now, for φ_0 and ψ in the condition ($\sharp$), we put $\mathfrak{M}(\varphi_0)=\{\varphi\in\mathfrak{M}(E);\ \varphi$ is equal to φ_0 on $E_0\}$ and define a map $\tilde{\sigma}: X\ni x\mapsto\tilde{\sigma}(x)=\sigma(x)^*\psi\in\mathfrak{M}(\varphi_0)$. For every $(s,\varphi)\in S\times\mathfrak{M}(\varphi_0)$ define $s\varphi\in\mathfrak{M}(\varphi_0)$ by $s\varphi=\varphi T_s$. Then $(S,\mathfrak{M}(\varphi_0))$ is a compact transformation semigroup and the map $\tilde{\sigma}$ is a homomorphism of (S,X) to $(S,\mathfrak{M}(\varphi_0))$. If (S,X) is extremely amenable, by Theorem 1, $(S,\mathfrak{M}(\varphi_0))$ has a fixed-point, that is, there exists a $\varphi\in\mathfrak{M}(E)$ such that $\varphi(T_s f)=\varphi(f)$ and $\varphi(g)=\varphi_0(g)$ for all $(s,f,g)\in S\times E\times E_0$. Thus if (S,X) is extremely amenable, any Banach algebra representation $\{T, E,\sigma,F\}$ of (S,X) has multiplicative invariant extension property. Conversely suppose that $\{L,B(X),\delta,R\}$ has multiplicative invariant extension property. Let $E_0=\{cI_X;\ c\in R\}$ and $\varphi_0\in\mathfrak{M}(E_0)$ define by $\varphi_0(cI_X)=c$. Then $\varphi_0({}_sf)=\varphi_0(f)=\psi(\delta(x)f)$ for all $(s,x,f)\in S\times X\times E_0$, where ψ is the identity map on R. So, by multiplicative invariant extension property, there exists a $\varphi\in\mathfrak{M}(B(X))=\beta X$ such that $\varphi({}_sf)=\varphi(f)$ for all $(s,f)\in S\times B(X)$. Thus we have

Theorem 2. *The following conditions are equivalent.*

(1) *A transformation semigroup* (S,X) *is extremely amenable.*

(2) *Any Banach algebra representation* $\{T,E,\sigma,F\}$ *of* (S,X) *has multiplicative invariant extension property.*

Corollary. *The following conditions are equivalent.*

(1) *A transformation semigroup* (S,X) *is extremely amenable.*

(2) *For any Banach algebra representation* $\{T,E,\sigma,F\}$ *of* (S,X) *if* $\mathfrak{M}(F)$ *is nonempty, then there is a* $\varphi\in\mathfrak{M}(E)$ *such that* $\varphi(T_s f)=\varphi(f)$ *for all* $(s,f)\in S\times E$.

§ 3. Reiter-Glicksberg's inequality.

Let $\{T,E,\sigma,F\}$ be an algebra representation of a transformation semigroup (S,X) and $H(T)$ the left ideal of E generated by $\{T_s v-v;\ v\in E, s\in S\}$.

Theorem 3. *The following conditions are equivalent.*

(1) *A transformation semigroup* (S,X) *is extremely amenable.*

(2) *For any algebra representation* $\{T,E,\sigma,F\}$ *of* (S,X) *we have* $H(T)\subseteq\{u\in E;\ \sigma(x)u=0$ *for some* $x\in SX\}$.

Proof. (1)$\Rightarrow$(2): Let $f=\sum_{i=1}^n u_i(T_{s_i}v_i-v_i)\in H(T)$ where $(u_i,v_i,s_i)\in E\times E\times S$ for $1\leq i\leq n$. Then, from Theorem 2 in [6], there is a point x_0 in SX such that $s_i x_0=x_0$ for $1\leq i\leq n$. So we have $\sigma(x_0)f=\sum_{i=1}^n\sigma(x_0)u_i(\sigma(s_i x_0)v_i-\sigma(x_0)v_i)=0$.

(2)$\Rightarrow$(1): Applying the condition (2) to the algebra representation $\{L,B(X),\delta,R\}$ of (S,X) for any function $h\in H(L)$ (which is denoted by $\mathfrak{H}(X)$ in [6]) there exists a point x_0 in X such that $h(x_0)=0$. Thus, by Theorem 2 in [6], (S,X) is extremely amenable. q.e.d.

Now we shall say a Banach algebra representation $\{T,E,\sigma,F\}$ of (S,X) is *uniformly bounded* if $\|T_s\|\leq 1$ and $\|\sigma(x)\|\leq 1$ for all $(s,x)\in S\times X$.

Theorem 4. *The following conditions are equivalent.*

(1) *A transformation semigroup* (S, X) *is extremely amenable.*

(2) *For any uniformly bounded Banach algebra representation* $\{T, E, \sigma, F\}$ *of* (S, X) *we have*

$$\inf \{\|f-h\| ;\ h \in H(T)\} \geqq \inf \{\|\sigma(x)f\| ;\ x \in SX\} \qquad \text{(RG)}$$

for all $f \in E$.

Proof. (1)$\Rightarrow$(2): Let $h \in H(T)$. Then, by Theorem 3, there is a point x_0 in SX such that $\sigma(x_0)h=0$. So $\|f-h\| \geqq \|\sigma(x_0)f-\sigma(x_0)h\| = \|\sigma(x_0)f\|$. Thus we have (RG) for all $f \in E$.

(2)$\Rightarrow$(1): Applying (RG) to $\{L, B(X), \delta, R\}$ we have $\inf\{\|I-h\|; h \in H(L)\}=1$. Thus, by Lemma 3 (c) in [2], (S, X) is extremely amenable. q.e.d.

The inequality (RG) is a generalization of the so-called *Reiter-Glicksberg's inequality* (see Expose n°2 in [1]).

§4. Addenda to my papers [6] and [7]. 1) In the proof of Theorem 1 in [6, p. 425], there is a gap. So we insert the following Lemma 2 after Lemma 1 in [6]. Then the proof of Theorem 1 in [6] becomes clear.

Lemma 2. *Under the notations in* [6, p. 425], *put* $Z=\bigcup_{k=2}^{\infty}(\bigcup_{i=1}^{k-1} X_i^k)$. *Then there exists a partition* $Z=\bigcup_{i=1}^{5} Z_i$ *such that* $sZ_i \cap Z_i = \phi$ *for* $1 \leq i \leq 5$.

2) In the statement of Theorem 1 (2) in [7], instead of "$\cdots \varphi \in IM(X) \cdots$" we should be "$\cdots \varphi \in \beta X \cdots$".

References

[1] P. Eymard: Moyennes invariantes et Representations unitaires. Lecture Notes in Math., Vol. 300. Springer-Verlag (1972).

[2] E. E. Granirer: Extremely amenable semigroups. II. Math. Scand., **20**, 93–113 (1967).

[3] ——: Functional analytic properties of extremely amenable semigroups. Trans. Amer. Math. Soc., **137**, 53–75 (1969).

[4] A. T. Lau: Functional analytic properties of topological semigroups and N-extremely amenability. ibid., **152**, 431–439 (1970).

[5] T. Mitchell: Fixed points and multiplicative invariant means. ibid., **122**, 195–202 (1966).

[6] K. Sakai: Extremely amenable transformation semigroups. Proc. Japan Acad., **49**, 424–427 (1973).

[7] ——: Amenable transformation groups. II. ibid., **49**, 428–431 (1973).