81. *Harmonic Analysis on Some Types of Semisimple Lie Groups*

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1. Introduction. Let G be a semisimple Lie group and let $L^2(G)$ denote the space of square integrable functions on G with respect to the Haar measure. The Fourier transform $\mathcal{F}$ can be regarded as an isometry of $L^2(G)$ onto a Hilbert space $L^2(\hat{G})$, which is defined with the help of some types irreducible unitary representations of G.

In his paper [5], Harish-Chandra introduces the Schwartz space $\mathcal{C}(G)$, of functions on G. It is analogous to the Schwartz space $\mathcal{S}(\boldsymbol{R}^n)$ ([6]), of rapidly decreasing functions on a euclidean space $\boldsymbol{R}^n$, and is contained densely in $L^2(G)$.

It is an interesting problem to characterize the image $\mathcal{C}(\hat{G})$ of $\mathcal{C}(G)$ in $L^2(\hat{G})$ by the Fourier transform $\mathcal{F}$. In the case that the real rank of G equals one, J. G. Arthur [1] solves this problem. Moreover, these problems for certain subspaces of $\mathcal{C}(G)$ are studied in the papers [3], [4] and [2].

The purpose of this paper is to give a characterization of the Fourier image $\mathcal{C}(\hat{G})$ for non-compact real semisimple Lie groups G with only one conjugacy class of Cartan subgroups.

The difficult part of this theorem is to prove surjectivity. For this, we must study in detail the asymptotic behaviour of the Eisenstein integrals, in particular, not only the constant terms but the asymptotic behaviour of them along the walls of Weyl chambers.

Detailed proofs will appear elsewhere.

2. Notation and preliminaries. Let $\boldsymbol{G}$ be a connected non-compact real semisimple Lie group with finite center. Let $\mathfrak{g}$ be the Lie algebra of G. Throughout this paper, we assume that G has only one conjugacy class of Cartan subgroups and $\mathfrak{g}$ does not have any complex structure. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a fixed Cartan decomposition with Cartan involution θ, $\mathfrak{a}_\mathfrak{p}$ a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{p}^*$ its dual space respectively. Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ which contains $\mathfrak{a}_\mathfrak{p}$ and put $\mathfrak{a}_\mathfrak{k}=\mathfrak{a}\cap\mathfrak{k}$. Let $\mathfrak{g}^c$ and $\mathfrak{a}^c$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{a}$ respectively, and Δ denote the set of non-zero roots of $\mathfrak{g}^c$ with respect to $\mathfrak{a}^c$. We introduce a linear order in Δ which is compatible with respect to $\mathfrak{a}_\mathfrak{p}$. Let Δ_+ and P_+ denote the set of positive roots and those which do not vanish on $\mathfrak{a}_\mathfrak{p}$. Let P_- denote the complement of P_+ in Δ_+. Define

ρ by $\rho=\frac{1}{2}\sum_{\alpha\in P_+}\alpha$ on $\mathfrak{a}_\mathfrak{p}$ and $\rho=0$ on $\mathfrak{a}_\mathfrak{p}$. We put $\mathfrak{n}=\left(\sum_{\alpha\in P_+}\mathfrak{g}^\alpha\right)\cap\mathfrak{g}$, where

$$\mathfrak{g}^\alpha=\{X\in\mathfrak{g}^c\mid[H,X]=\alpha(H)X \qquad \textit{for all } H\in\mathfrak{a}^c\}.$$

For each $\alpha\in\Delta_+$ define an element H_α in $\mathfrak{a}^c$ by $B(H_\alpha,H)=\alpha(H)$, for all $H\in\mathfrak{a}^c$, where B is the Killing form of $\mathfrak{g}^c$ restricted to $\mathfrak{a}^c$. Put $\varpi=\prod_{\alpha\in\Delta_+}H_\alpha$ and $\varpi_m=\prod_{\alpha\in P_-}H_\alpha$, which can be polynomial functions on $\mathfrak{a}^c$. Let K, A_+ and N be the analytic subgroups of G corresponding to $\mathfrak{k}$, $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{n}$ respectively. Let M and M' be the centralizer and normalizer of $\mathfrak{a}_\mathfrak{p}$ in K respectively. The finite group $W=M'/M$ is called the Weyl group. Let A be the centralizer of $\mathfrak{a}$ in G and put $A_-=A\cap K$. Then A and A_- are Cartan subgroups of G and the compact reductive group M respectively. Put $P=MA_+N$, which is a minimal parabolic subgroup. Let Π be the lattice of linear functionals

$$\mu:(-1)^{\frac{1}{2}}\mathfrak{a}_\mathfrak{t}\to\boldsymbol{R}$$

for which $\xi_\mu(\exp H)=\exp\{\mu(H)\}$ $(H\in\mathfrak{a}_\mathfrak{t})$ gives a character ξ_μ of A_-. We put $\Pi'=\{\mu\in\Pi\mid\mu(H_\alpha)\neq 0$ for every $\alpha\in P_-\}$. Let $\mathcal{E}_M$ be the set of equivalence classes of irreducible unitary representations of M. Then $\mathcal{E}_M$ can be indexed by Π'. If $\sigma\in\mathcal{E}_M$ and $\sigma=\sigma(\mu)$ for $\mu\in\Pi'$, we write $|\sigma|^2=B(\mu,\mu)$.

Let $\mathcal{E}_K$ denote the set of unitary equivalence classes of irreducible unitary representations of K. It is known that the representations in $\mathcal{E}_K$ can be also indexed by real linear functionals in Π'. If $\tau\in\mathcal{E}_K$ and $\tau=\tau(\nu)$ for $\nu\in\Pi'$, put $|\tau|^2=B(\nu,\nu)$.

If $\sigma\in\mathcal{E}_M$ acts on the finite dimensional Hilbert space V_σ and $\lambda\in\mathfrak{a}_\mathfrak{p}^*$,

$$\pi_{\sigma,\lambda}=\operatorname*{Ind}_{P\uparrow G}\sigma\lambda,$$

where $\sigma\lambda$ is given by

$$(\sigma\lambda)(man)=\sigma(m)\exp\{-i\lambda(\log a)\},\quad m\in M,\quad a\in A_+,\quad n\in N,$$

are the induced representations. $\pi_{\sigma,\lambda}$ is a unitary representation of G on a Hilbert space $\mathcal{H}_{\sigma,\lambda}$, which is the set of functions Φ from G into V_σ which satisfy the following conditions: (i) $\Phi(x\xi^{-1})=(\sigma\lambda)(\xi)\Phi(x)$, $x\in G$, $\xi\in P$, (ii) $\Phi(k)$ is a Borel function on K, (iii) $\int_K|\Phi(k)|^2\,dk<+\infty$. The inner product on $\mathcal{H}_{\sigma,\lambda}$ is given by

$$(\Phi,\Psi)=\int_K(\Phi(k),\Psi(k))_{V_\sigma}dk,\qquad \Phi,\Psi\in\mathcal{H}_{\sigma,\lambda},$$

where $(\ ,\ )_{V_\sigma}$ is the inner product in V_σ. For $\Phi\in\mathcal{H}_{\sigma,\lambda}$ $\pi_{\sigma,\lambda}\Phi$ is defined by

$$(\pi_{\sigma,\lambda}(y)\Phi)(x)=\Phi(y^{-1}x)\exp\{-\rho(H(y^{-1}x))+\rho(H(x))\},\qquad x,y\in G.$$

For any $\lambda\in\mathfrak{a}_\mathfrak{p}^*$ and any $\Phi\in\mathcal{H}_{\sigma,\lambda}$ put $\tilde{\Phi}=\Phi|K$, the restriction of Φ to K. This identifies $\mathcal{H}_{\sigma,\lambda}$ with a Hilbert space $\mathcal{H}_\sigma$, of square-integrable functions from K into V_σ.

Each $s\in W$ induces an automorphism of M, modulo the group of inner automorphisms. Hence s defines the bijection $s:\sigma\to s\sigma$ of $\mathcal{E}_M$

onto itself. It is known that the representation $\pi_{\sigma,\lambda}$ is equivalent to $\pi_{s\sigma,s\lambda}$ if $\lambda \in \mathfrak{a}_{\mathfrak{p}}^*$ is regular. We write $\mathfrak{a}_{\mathfrak{p}}^{*\prime}$ as the set of regular elements in $\mathfrak{a}_{\mathfrak{p}}^*$. For each $\sigma \in \mathcal{E}_M$ and $\lambda \in \mathfrak{a}_{\mathfrak{p}}^{*\prime}$, let $N_\sigma^s(\lambda)$ be a fixed unitary intertwining operator between the representations $\pi_{\sigma,\lambda}$ and $\pi_{s\sigma,s\lambda}$. Then

$$N_\sigma^s(\lambda)\ \pi_{\sigma,\lambda}(x) N_\sigma^s(\lambda)^{-1} = \pi_{s\sigma,s\lambda}(x), \qquad x \in G.$$

3. The Plancherel formula. For any $\mu \in \Pi'$ and $\lambda \in \mathfrak{a}_{\mathfrak{p}}^*$, we put

$$\varpi(\mu, \lambda) = \varpi(\mu + i\lambda).$$

If $\sigma \in \mathcal{E}_M$ and $\lambda \in \mathfrak{a}_{\mathfrak{p}}^*$ let $\Theta_{\sigma,\lambda}$ be the character of the representation $\pi_{\sigma,\lambda}$.

Lemma. *We can select a non-negative function $\beta(\sigma, \lambda)$ on $\mathcal{E}_M \times \mathfrak{a}_{\mathfrak{p}}^*$ and a constant c such that*

$$\beta(\sigma, \lambda) = c \cdot \varpi(\mu, \lambda), \quad \sigma = \sigma(\mu) \in \mathcal{E}_M, \quad \lambda \in \mathfrak{a}_{\mathfrak{p}}^*,$$

and

$$f(1) = \sum_{\sigma \in \mathcal{E}_M} \int_{\mathfrak{a}_{\mathfrak{p}}^{*+}} \beta(\sigma, \lambda) \Theta_{\sigma,\lambda}(f) d\lambda, \qquad f \in C_c^\infty(G),$$

where $\mathfrak{a}_{\mathfrak{p}}^{*+}$ is the positive Weyl chamber and $d\lambda$ is a normalized measure on $\mathfrak{a}_{\mathfrak{p}}^*$.

For $h_- \in A_-$, $h_+ \in A_+$ and $h = h_- h_+$ a regular element in A, put

$$\Delta(h) = \xi_\rho(h) \prod_{\alpha \in \Delta_+} (1 - \xi_\alpha(h)^{-1}),$$

where ξ_ρ and ξ_α are the characters on A such that $\xi_\rho(\exp H) = \exp\{\rho(H)\}$, $\xi_\alpha(\exp H) = \exp\{\alpha(H)\}$, $H \in \mathfrak{a}^c$. Put $m = \frac{1}{2}(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})$. Let G^* denote the homogeneous space G/A and dx^* the G-invariant measure on G^*. For $f \in C_c^\infty(G)$, put

$$F_f(h) = \Delta(h) \int_{G^*} f(x^* h x^{*-1}) dx^*, \qquad h \in A',$$

where A' denotes the set of regular elements in A.

Theorem 1. *There is a positive constant c_0 such that for any $f \in C_c^\infty(G)$,*

$$\Theta_{\sigma,\lambda}(f) = c_0(-1)^m (\operatorname{sign} \varpi^m(\mu)) \int_{A_- \times A_+} F_f(h_- h_+) \xi_\mu(h_-) \exp\{-i\lambda(\log h_+)\} dh_- dh_+,$$

where dh_- and dh_+ denote normalized Haar measures on A_- and A_+ respectively.

For each $\sigma \in \mathcal{E}_M$, $\mathcal{H}_2(\sigma)$ denotes the space of Hilbert-Schmidt operators on $\mathcal{H}_\sigma$ with the Hilbert-Schmidt norm $\|\cdot\|_2$.

Let $L^2(\hat{G})$ be the set of functions

$$a : \mathcal{E}_M \times \mathfrak{a}_{\mathfrak{p}}^* \to \bigoplus_{\sigma \in \mathcal{E}_M} \mathcal{H}_2(\sigma)$$

satisfying the following conditions (i), (ii), (iii) and (iv):

(i) $a(\sigma, \lambda) \in \mathcal{H}_2(\sigma)$ for each $\sigma \in \mathcal{E}_M$, $\lambda \in \mathfrak{a}_{\mathfrak{p}}^*$,

(ii) $a(s\sigma, s\lambda) = N_\sigma^s(\lambda) a(\sigma, \lambda) N_\sigma^s(\lambda)^{-1}$, $\sigma \in \mathcal{E}_M$, $\lambda \in \mathfrak{a}_{\mathfrak{p}}^{*\prime}$, $s \in W$,

(iii) For any $\sigma \in \mathcal{E}_M$, $a(\sigma, \lambda)$ is a Borel function of λ,

(iv) $\|a\|^2=\frac{1}{\omega}\sum_{\sigma\in\mathcal{E}_M}\int_{\mathfrak{a}_\mathfrak{p}^*}\|a(\sigma,\lambda)\|_2^2\,\beta(\sigma,\lambda)d\lambda<\infty$,

where ω denotes the order of W.

Then $L^2(\hat{G})$ is a Hilbert space. If $f\in C_c^\infty(G)$, define $\hat{f}\in L^2(\hat{G})$ by

$$\hat{f}(\sigma,\lambda)=\int_G f(x)\pi_{\sigma,\lambda}(x)dx,\quad \sigma\in\mathcal{E}_M,\quad \lambda\in\mathfrak{a}_\mathfrak{p}^*.$$

We call the map $\mathcal{F}: f\to\hat{f}$ of $C_c^\infty(G)$ into $L^2(\hat{G})$ the Fourier transform.

Theorem 2 (Plancherel formula). *The Fourier transform*

$$\mathcal{F}: f\to\hat{f},\qquad f\in C_c^\infty(G),$$

extends uniquely to an isometry of $L^2(G)$ onto $L^2(\hat{G})$.

4. The Schwartz space. For $x\in G$, define

$$\Xi(x)=\int_K \exp\{-\rho(H(xk))\}dk.$$

Since $G=KA_+K$ there exists a unique function on G such that

(i) $\sigma(k_1xk_2)=\sigma(x)$, $k_1,k_2\in K$, $x\in G$,

(ii) $\sigma(\exp H)=B(H,H)^{\frac{1}{2}}$, $H\in\mathfrak{a}_\mathfrak{p}$.

For every g_1, g_2 in $\mathcal{B}$ and s in $\boldsymbol{R}$, we define a semi-norm on $C^\infty(G)$ by

$$|f|_{g_1,g_2,s}=\sup_{x\in G}|f(g_1;x;g_2)|\,\Xi(x)^{-1}(1+\sigma(x))^s,\qquad f\in C^\infty(G).$$

Let $\mathcal{C}(G)=\{f\in C^\infty(G)\,|\,|f|_{g_1,g_2,s}<\infty$, for any g_1,g_2 in $\mathcal{B}$ and s in $\boldsymbol{R}\}$. These semi-norms make $\mathcal{C}(G)$ into a Fréchet space. $\mathcal{C}(G)$ is called the Schwartz space of G. It is known that the inclusions

$$C_c^\infty(G)\subset\mathcal{C}(G)\subset L^2(G)$$

are continuous with respect to the usual topologies.

5. The main theorems. Fix $\tau\in\mathcal{E}_K$ and $\sigma\in\mathcal{E}_M$ acting on the Hilbert spaces V_τ and V_σ of dimension t and s respectively. Let $R(\tau,\sigma)$ be the set of intertwining operators from V_τ to V_σ for $\tau|M$ and σ, where $\tau|M$ is the restriction of τ to M. The Hilbert-Schmidt norm makes $R(\tau,\sigma)$ into a Hilbert space of dimension $[\tau:\sigma]$, where $[\tau:\sigma]$ denotes the multiplicity of σ in $\tau|M$.

Let $\{\xi_1,\cdots,\xi_t\}$ be a fixed orthonormal base of V_τ. We write $\tau^*(k)$ for $\tau(k^{-1})$ if $k\in K$. Fix an orthonormal base $\{T_1,\cdots,T_r\}$ of $R(\tau,\sigma)$ of elements of norm equal to $(\dim\tau)^{\frac{1}{2}}$. For $1\leqq l\leqq r$, $1\leqq j\leqq t$, and $k\in K$, define

$$\Phi_{\tau,(l-1)t+j}(k)=T_l(\tau^*(k)\xi_j).$$

Then $\{\Phi_{\tau,i}\,|\,\tau\in\mathcal{E}_K,\ 1\leqq i\leqq[\tau:\sigma]\cdot\dim\tau\}$ is an orthonormal base for $\mathcal{H}_\sigma$.

Let $\mathcal{C}(\hat{G})$ denote the set of functions $a(\sigma,\lambda)$ of $\mathcal{E}_M\times\mathfrak{a}_\mathfrak{p}^*$ into $\mathcal{H}_2(\sigma)$ which satisfy the following conditions (i),(ii) and (iii):

(i) For each $\sigma\in\mathcal{E}_M$, $a(\sigma,\lambda)$ is a matrix valued C^∞ function on $\mathfrak{a}_\mathfrak{p}^*$,

(ii) $a(s\sigma,s\lambda)=N_\sigma^s(\lambda)a(\sigma,\lambda)N_\sigma^s(\lambda)^{-1}$, $\sigma\in\mathcal{E}_M$, $\lambda\in\mathfrak{a}_\mathfrak{p}^{*\prime}$, $s\in W$,

(iii) For every set of polynomials (p_1,p_2,q_1,q_2) and each $d\in\boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^*)$, where $\boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^*)$ is the algebra of differential operators on $\mathfrak{a}_\mathfrak{p}^*$ with constant coefficients,

$$|a|_{(p_1,p_2,q_1,q_2:d)} = |d_\lambda(\Phi_{\tau_1,i_1}, a(\sigma,\lambda)\Phi_{\tau_2,i_2})| p_1(|\sigma|)p_2(|\lambda|)q_1(|\tau_1|)q_2(|\tau_2|) < +\infty.$$

Then the above semi-norms define a topology on $\mathcal{C}(\hat{G})$ so that $\mathcal{C}(\hat{G})$ is a Fréchet space. $\mathcal{C}(\hat{G})$ is contained densely in $L^2(\hat{G})$.

Theorem 3. *The Fourier transform*

$$\mathcal{F}: f \to \hat{f}$$

is a topological isomorphism of $\mathcal{C}(G)$ onto $\mathcal{C}(\hat{G})$.

If we consider the strong duals $\mathcal{C}'(G)$ and $\mathcal{C}'(\hat{G})$, of the Fréchet spaces $\mathcal{C}(G)$ and $\mathcal{C}(\hat{G})$ respectively, and the transposed inverse $(\mathcal{F}^{-1})^*$ of the Fourier transform $\mathcal{F}$, as a corollary of the theorem, we obtain the following

Theorem 4. *The map $(\mathcal{F}^{-1})^*$ is a topological isomorphism of $\mathcal{C}'(G)$ onto $\mathcal{C}'(\hat{G})$.*

References

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